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The D-instanton partition function

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Abstract: The D-instanton partition function is a fascinating quantity because in the presence of N D3-branes, and in a certain decoupling limit, it reduces to the functional integral of $\mathcal{N}=4$ $\mathrm{U}(N)$ supersymmetric gauge theory for multi-instanton solutions. We study this quantity as a function of non-commutativity in the D3-brane theory, VEVs corresponding to separating the D3-branes and α'. Explicit calculations are presented in the one-instanton sector with arbitrary N, and in the large-N limit for all instanton charge. We find that for general instanton charge, the matrix theory admits a nilpotent fermionic symmetry and that the action is $\mathcal{Q}$-exact. Consequently the partition function localizes on the minima of the matrix theory action. This allows us to prove some general properties of these integrals. In the non-commutative theory, the contributions come from the "Higgs Branch" and are equal to the Gauss-Bonnet-Chern integral of the resolved instanton moduli space. Separating the D3-branes leads to additional localizations on products of abelian instanton moduli spaces. In the commutative theory, there are additional contributions from the "Coulomb Branch" associated to the small instanton singularities of the instanton moduli space. We also argue that both non-commutativity and α'-corrections play a similar rôle in suppressing the contributions from these singularities. Finally we elucidate the relation between the partition function and the Euler characteristic of the instanton moduli space.

Keywords: Solitons Monopoles and Instantons, D-branes, Extended Supersymmetry.

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Contents

1. Instantons and D-branes

One of the most fascinating recent developments has been the rapprochement of string theory and Yang Mills theory. Supersymmetric versions of the latter (SYM) naturally arise as the low energy collective dynamics of D-brane solitons in type-II string theory. This point-of-view turns out to be useful in both directions: both string theory and SYM benefit. In this paper, we will be considering the interplay of instantons in SYM and their description in string theory as Dp-branes bound to D$(p+4)$-branes. It turns out that the instanton calculus of $\mathcal{N}=4$ SYM_4 can be derived in a relatively painless way by considering the dynamics of such a brane system. In particular, the rather mysterious ADHM construction of multi-instantons [1, 2] is recovered in a very elegant way [5, 6, 7]. However, the relation is much more far-reaching than just recovering the moduli space of instantons: remarkably, the multi-instanton integration measure that arises from changing variables in the functional integral of SYM_4 to the instanton collective coordinates [3, 4], is precisely the partition function of a system of D-instantons moving in the background of D3-branes in a certain decoupling limit [4]–[8]. The central theme of this paper will be to investigate properties of this partition function along with some explicit evaluations. The underlying motive being to provide new tools for calculating various multi-instanton effects in SYM and we believe that the ideas of topological field theory in the context of these ADHM matrix integrals will prove to be very powerful.

The low energy collective dynamics of a collection of N coincident D$(p+4)$-branes in type-II string theory is described by $(p+5)$-dimensional U(N) SYM_{p+5} with 16 real supercharges. For $p \geq -1$, we can naturally embed a gauge theory multi-instanton solution on the world-volume. This solution will be some co-dimension-four multi-soliton, i.e. a p-brane extended in $p+1$ spacetime dimensions. The crucial fact due to Witten [5] and Douglas [6, 7] is that when a instanton shrinks to zero size, it is precisely a Dp-brane lying in the world-volume of the D$(p+4)$-branes. For instance, it is easy to argue that the Dp-brane carries a unit of instanton charge of the gauge field of the higher-dimensional brane.

This description of Yang-Mills instantons has far-reaching consequences because by shifting our attention to the dynamics of the Dp-branes themselves we actually arrive at the calculus of SYM instantons in a straightforward way. In the absence of the D$(p+4)$-branes, the low energy collective dynamics of k Dp-branes is described by U(k) SYM_{p+1} with 16 real supercharges. The adjoint-valued fields are associated to open strings which begin and end on the k branes. The theory is simply the dimensional reduction to $(p+1)$-dimensional spacetime of $\mathcal{N}=1$ SYM_{10}. The ten-dimensional gauge field becomes a gauge field in $(p+1)$-dimensions along with $9-p$ adjoint scalar fields. The resulting theory has a $(9-p)$-dimensional Coulomb branch on which the $9-p$ adjoint scalars gain a VEV. Up to U(k), the Coulomb branch is described by the diagonal elements of the adjoint scalars which specify the position of the Dp-branes in the space transverse to their world-volume. When we add the N D$(p+4)$-branes, coincident to begin with, there are N additional U(k)-fundamental hypermultiplets which break half the supersymmetries, so the resulting theory has 8 real supercharges. These additional fields correspond to open strings stretched between the two types of branes.

The Dp/D$(p+4)$-brane system can live in maximal dimension $p=5$, corresponding to an $\mathcal{N}=(1,0)$ SYM_6 on the world volume of the D5-branes. However, we will find it more convenient to use the more familiar language of $\mathcal{N}=2$ and $\mathcal{N}=1$ superfields in the four dimensional, $p=3$, case. In this language, there is a vector multiplet of $\mathcal{N}=2$ which decomposes as a vector multiplet of $\mathcal{N}=1$ and a chiral multiplet Φ. The complex scalar $\phi \subset \Phi$, along with $3-p$ components of the four-dimensional gauge field, describe the positions of the Dp-branes in the $(5-p)$-dimensional space orthogonal to the D$(p+4)$-branes. There is an U(k)-adjoint hypermultiplet which is composed of two chiral multiplets X and $\tilde{X}$ which describe the positions of the Dp-branes along the D$(p+4)$-branes. Finally there are N U(k)-fundamental hypermultiplets which are composed of 2 chiral multiplets Q and $\tilde{Q}$. The theory has a U(N) flavour symmetry which acts as $Q \to QU$, $\tilde{Q} \to U^\dagger \tilde{Q}$.

The lagrangian of the theory is schematically of the form

$$\mathcal{L} = g_{p+1}^{-2}\mathcal{L}_{V,\Phi} + \mathcal{L}_{X,\tilde{X}} + \mathcal{L}_{Q,\tilde{Q}}\,, \tag{1.1}$$

where $\mathcal{L}_{V,\Phi}$ is the lagrangian for the the vector multiplet, while $\mathcal{L}_{X,\tilde{X}}$ and $\mathcal{L}_{Q,\tilde{Q}}$ are the lagrangians describing the hypermultiplets and their coupling to the vector multiplet. The vector multiplet involves the dimensionful coupling constant $g_{p+1}^2 = 2(2\pi)^{p-2}e^{\phi}\alpha'^{(p-3)/2}$.

Let us now analyse the classical phase structure of this theory. As usual in theories with 8 supercharges, there is a moduli space of vacua described by the vanishing of the F- and D-terms. The equations for the vacuum depends upon the number of spacetime dimensions $p+1$. For $p=3$, for example, we have the F-term equations

$$q\tilde{q} + [x,\tilde{x}] = [\phi,x] = [\phi,\tilde{x}] = \phi q = \tilde{q}\phi = 0\,, \tag{1.2}$$

and the real D-term equation

$$qq^\dagger - \tilde{q}^\dagger\tilde{q} + [x,x^\dagger] + [\tilde{x},\tilde{x}^\dagger] + [\phi,\phi^\dagger] = 0\,. \tag{1.3}$$

For $p<3$, the scalars that arise from the dimensional reduction of the gauge field can also get VEVs and in this case the equations are more complicated. However, there is one branch of solutions, the "Higgs branch", which is independent of p. On this branch only the hypermultiplet VEVs are non vanishing, and so

$$q\tilde{q} + [x,\tilde{x}] = 0\,, \qquad qq^\dagger - \tilde{q}^\dagger\tilde{q} + [x,x^\dagger] + [\tilde{x},\tilde{x}^\dagger] = 0\,. \tag{1.4}$$

These comprise $3k^2$ real equations for the $4k(N+k)$ unknowns q, $\tilde{q}$, x and $\tilde{x}$. Therefore, up to the $\mathrm{U}(k)$ gauge symmetry, they describe a $4kN$-dimensional moduli space $\mathfrak{M}_{k,N}$ which is guaranteed to be hyper-Kähler, since it is the Higgs branch of a SYM theory with 8 real supercharges. In fact the left-hand sides of (1.4) are nothing but the moment maps of the hyper-Kähler quotient construction [9].

The crucial fact is that $\mathfrak{M}_{k,N}$ *is* the moduli space of k instantons in $\mathrm{U}(N)$ gauge theory as constructed by ADHM. Eqs. (1.4) are the celebrated ADHM constraints [1, 10, 11]. This derivation of the ADHM construction is rather satisfying because the $\mathrm{U}(k)$ auxiliary symmetry of the ADHM construction arises as a conventional gauge symmetry and the mysterious ADHM collective coordinates are nothing but the VEVs of the scalars in the hypermultiplets. What is missing is the actual construction of the self-dual gauge potential, however, this may also be derived by considering a "probe" brane moving in the background of the Dp/D$(p+4)$-brane system [12].

The interpretation of the VEVs as the collective coordinates of instantons becomes apparent in the clustering regime where the latter are well separated. In this case, up to U(k) the $k \times k$ matrices x and $\tilde{x}$ are approximately diagonal with eigenvalues x_i and $\tilde{x}_i$, $i = 1, \ldots, k$. The k four-vectors

$$\Big(\operatorname{Re} x_i, \operatorname{Im}\, x_i, \operatorname{Re} \tilde{x}_i, \operatorname{Im} \tilde{x}_i\Big)\,, \tag{1.5}$$

give the positions of k separated instantons in the $\mathbb{R}^4$ transverse to the Dp-branes in the D$(p+4)$-branes. In this clustering region of the moduli space, the diagonal components of the $k \times k$ matrix

$$\rho_i^2 = \frac{1}{2}\left(qq^\dagger + \tilde{q}^\dagger \tilde{q}\right)_{ii}\,, \tag{1.6}$$

give the instanton scale sizes ρ_i and finally the 3 $N \times N$ matrices

$$T_i^3 = q_i^\dagger q_i - \tilde{q}_i \tilde{q}_i^\dagger\,, \qquad T_i^+ = \tilde{q}_i q_i\,, \qquad T_i^- = q_i^\dagger \tilde{q}_i^\dagger\,, \tag{1.7}$$

describe the SU(2) orientation of the i-th instanton in the U(N) gauge group.

Since the VEVs of Φ, and any additional scalars from the vector multiplet, vanish, the Higgs branch describes a situation where the Dp-branes are "dissolved" in the D$(p+4)$-branes and fattened out.[1] Generically U(k) is completely broken by the VEVs[2] of q and $\tilde{q}$. However, when one of the instantons shrinks to zero size the corresponding components of the fundamental hypermultiplets vanish: $q_i = \tilde{q}_i = 0$. At these singular points, whose nature we will elucidate in more detail below, the corresponding components of the chiral field ϕ_{ii} (and the corresponding scalars that arise from the gauge field if $p < 3$) can become non zero, as is evident from (1.2). This describes a situation in which the instanton that has shrunk to zero size, can move off as Dp-branes into the bulk space transverse to the D$(p+4)$-branes. In this case U(k) (generically) is only broken to U(1). The situation where a subset of the instantons move off into the bulk describes a "mixed branch" of the vacuum moduli space. When all the instantons move off into the bulk, $q = \tilde{q} = 0$, the gauge group is generically broken down to $U(1)^k$. This is the "Coulomb branch" of the vacuum moduli space.

Classically the phases are connected at the singular points where instantons shrink to zero size and the solution of the ADHM equations (1.4) is fixed by a subgroup of U(k). In a technical sense, the ADHM moduli space of instantons excludes these points where U(k) does not act freely, and understanding what happens at these points will be one of the themes of this paper.

[1]The separations between the Dp-branes and the D$(p+4)$-branes are given by the masses of the bi-fundamental hypermultiplets Q and $\tilde{Q}$ times α'. In general, these masses can be induced by the VEVs of ϕ (and some components of the gauge field for $p < 3$). Since on the Higgs branch $\phi = 0$, the bi-fundamental masses are zero and the separations between the Dp-branes and the D$(p+4)$-branes vanish.

[2]The case $N = 1$ is somewhat special as we will describe later.

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For the case with $p > -1$, so far all that have said is purely classical and we must consider how the picture is modified for the in the quantum theory. We will ultimately be interested in what happens in the D(-1)/D3-system where the theory of the instantons is a matrix theory (the "D-instanton matrix theory") because this will describe instanton effects in $\mathcal{N} = 4$ SYM_4 on the D3-branes. However, rather than proceed directly to this case, it will prove useful, en route, to visit the D0/D4- and D1/D5-systems which play an important rôle in the discrete light-cone quantization (DLCQ) of the $(2, 0)$ six-dimensional theories of M5-branes [13, 14] and NS5-branes, respectively. These latter theories are the "little string theories" recently reviewed in [15].

In the cases that we are focusing on, the world-volume theories of the Dp-branes have spacetimes of dimension 2 or smaller. In these cases, there cannot be any genuine moduli spaces of vacua due to the Mermin-Wagner theorem. Strong infrared fluctuations of the massless modes occur and the description of the physics in terms of the classical branches with symmetry breaking is not valid. What happens is that the wavefunctions spread out over the classical moduli space. Remarkably, it turns out that there are still distinct phases in the quantum theory which are related to the classical branches. The classical branches appear as the target spaces of σ-models that describe the theory at low energy. For example, the Higgs and Coulomb branches correspond to distinct quantum phases and are described by σ-models whose target spaces are the associated classical branches. Classically the Higgs and Coulomb branches touch at the point where instantons shrink to zero size, however, in the quantum theory the relation between the phases becomes more interesting since the points at which the classical phases touch are points where the σ-model description apparently breaks down because additional states become massless. In the D1/D5 system, the picture that has emerged is that in the quantum theory the singularities are replaced by semi-infinite throats and the phases become disconnected at low energy [16, 18]. The story in the D0/D4 system is similarly very interesting [17].

The dimensional reduction to $(0+0)$-dimensions of the theories described above, gives the D(-1)/D3-brane system. This encapsulates the instanton calculus of $\mathcal{N} = 4$ SYM_4, or, depending on whether we take the decoupling limit, D-instantons effects on D3-branes. In this case, the system is simply a matrix model and physical quantities are just finite-dimensional integrals over the matrix variables. At first sight, therefore, it is not clear whether the notion of phases has any rôle to play in this situation. One of the main themes of this paper is that, just as in the higher-dimensional cases, it is very useful to have in mind the concept of the phases due to a localization property of the matrix integral. The explanation is rather familiar: there is a nilpotent fermionic symmetry and the "action" of the matrix theory is $\mathcal{Q}$-exact and so the integral can be localized around the zeros of the action. The latter correspond to the phases of the higher-dimensional cases. This localization promises to be a new and powerful tool for calculating various multi-instanton effects in gauge theories. We will primarily be interested in calculating the D-instanton partition function which appears as an 8-fermion vertex of the D3-brane or, in the decoupling limit $g_0 = \infty$, of the $\mathcal{N} = 4$ SYM_4 effective action. In particular, we would like to determine how it depends on on three effects:

(i) VEVs φ_a, $a = 1, \ldots, 6$, in the D3-brane theory:

$$(\varphi_a)_{uv} = \varphi_{au}\delta_{uv} \,. \tag{1.8}$$

Physically this corresponds to the D3-branes separating in six-dimensional space transverse to their world-volume. In the D-instanton matrix theory this effect corresponds to introducing masses for the fundamental hypermultiplets.

(ii) Non-commutativity in the D3-brane theory, or alternatively turning on a background spacetime B-field. It turns out that considering the D3-brane theory on non-commutative spacetime [19, 20]

$$[x_n, x_m] = -i\zeta^c\bar{\eta}^c_{nm} \,, \tag{1.9}$$

where $\bar{\eta}^c_{nm}$ is a 't Hooft symbol, and the x_n are the four-dimensional spacetime coordinates (not to be confused with the quantities x and $\tilde{x}$ from the D-instanton matrix theory), corresponds in the instanton matrix theory to turning on Fayet Illiopolos (FI) couplings in the U(1) subgroup of the U(k) gauge group. We will define $\zeta_{\mathbb{R}} \equiv \zeta^3$ and $\zeta_{\mathbb{C}} \equiv \zeta^1 + i\zeta^2$.

(iii) String corrections. The D-instanton theory depends on the coupling $g_0 \sim e^{\phi}(\alpha')^{-2}$ via the kinetic term for the vector multiplet (1.1). In order to decouple string effects, and recover instanton calculus in $\mathcal{N} = 4$ SYM_4, we need to take the decoupling limit $\alpha' \to 0$ with fixed coupling on the D3-branes: so fixed $g_4 \sim e^{\phi}$. Hence, we must take $g_0 = \infty$. However, in certain circumstances, for instance when we want to calculate instanton effects in D3-branes (rather than their low energy limits), as in [21], then we need to include the g_0^{-2} couplings in order to break superconformal invariance.

We will denote the D-instanton partition function as

$$\mathcal{Z}_{k,N}(\zeta, g_0, \varphi) = \int dV\, d\Phi\, dX\, d\tilde{X}\, dQ\, d\tilde{Q}\, e^{-\mathcal{L}} \,. \tag{1.10}$$

This is not quite what we want because, as defined above, it vanishes because a given D-instanton configuration breaks half the supersymmetries of the D3-brane configuration and so there are 8 exact fermion zero modes — the goldstino modes of the broken supersymmetry — of the background. Associated to these modes are 8 Grassmann collective coordinates which are the superpartners of the "centre of mass" (COM) coordinates, $\mathrm{tr}_k\, x$ and $\mathrm{tr}_k\, \tilde{x}$, of the instanton configuration in $\mathbb{R}^4$. We can factor out the COM by removing the trace of the adjoint hypermultiplet $\{X, \tilde{X}\}$. This defines the "centered" partition function $\widehat{\mathcal{Z}}_{k,N}(\zeta, g_0, \varphi)$.

When $g_0 = \infty$ and there are no VEVs, the partition function $\widehat{\mathfrak{Z}}_{k,N}(0,\infty,0)$, reduces to an integral over the (centered) instanton moduli space $\widehat{\mathfrak{M}}_{k,N} = \mathfrak{M}_{k,N}/\mathbb{R}^4$. This is because in this limit the three auxiliary fields of the vector multiplet act as Lagrange multipliers for the ADHM constraints, while the gauge field can be integrated out via its equation-of-motion. Finally, the vector multiplet fermions act as fermionic Lagrange multiplers for the fsuperpartners of the ADHM constraints. So we can think of the partition function as an integral over the ADHM-instanton matrix theory. However, even though we have removed the COM degrees-of-freedom, the integral is still formally zero, since there is nothing to saturate the integrals over the 8 Grassmann collective coordinates associated to the broken superconformal invariance. In order to break superconformal invariance we have to modify the theory in some way. Each of the deformations (i)–(iii) described above introduces a scale into the problem and explicitly breaks superconformal invariance and renders the partition function well defined. For example, let us turn on non-commutativity, by taking non-trivial FI couplings. In that case, after integrating out the vector multiplet, the partition function $\widetilde{\mathfrak{Z}}_{k,N}(\zeta,\infty,0)$ reduces to an integral over a deformation of the instanton moduli space that we denote $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$. To see this, we note that the FI terms couple to the U(1) components of the vector multiplet and this modifies the D-term equations on the Higgs branch (1.4) to

$$q\tilde{q} + [x,\tilde{x}] = \zeta_{\mathbb{C}} 1_{[k]\times[k]}\,, \qquad qq^\dagger - \tilde{q}^\dagger\tilde{q} + [x,x^\dagger] + [\tilde{x},\tilde{x}^\dagger] = \zeta_{\mathbb{R}} 1_{[k]\times[k]}\,. \tag{1.11}$$

These equations, modulo U(k), describe the deformed, but still hyper-Kähler, moduli space $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$. The FI couplings have the effect of resolving, or blowing up, the small instanton singularities of $\widehat{\mathfrak{M}}_{k,N}$. The point is that because of the terms on the right-hand sides of (1.11), no components q_i and $\tilde{q}_i$ can vanish and instantons can only shrink to a minimal non-zero size which depends on the scale of the FI couplings.

The partition function $\widehat{\mathfrak{Z}}_{k,N}(\zeta,\infty,0)$ is now non zero since the FI couplings break superconformal invariance and the integral over the 8 Grassmann collective coordinates corresponding to broken superconformal invariance are now saturated. This partition function has the topological interpretation as the Gauss-Bonnet-Chern (GBC) integral over the space $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$ or, equivalently, the bulk contribution to the $\mathcal{L}^2$-index of harmonic forms. It is also fruitful to view it as the bulk contribution to the Witten index of the supersymmetric quantum mechanical system on $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$. One way to see this is to relate the D-instanton partition function, via T-duality, to the quantum mechanical gauge theory of the D0/D4-brane system. The low energy description of the latter is a quantum mechanical σ-model with a target space $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$. The reason why we only get the bulk contribution to the index is because the target space $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$, although smooth, is non compact, since instantons can separate in $\mathbb{R}^4$ and become arbitrarily large. In other words the quantum mechanical system has a potential with flat directions. The main problem introduced by non compactness is the fact that the theory has a continuous spectrum of scattering states in addition to the discrete bound state spectrum. In particular, even scattering states of non-zero energy can actually contribute to the Witten index. Naïvely states of non-zero energy come in bose-fermi pairs which cancel in the Witten index due to the insertion of $(-1)^F$ appearing in the trace. However, although supersymmetry demands that the range of the continuous spectrum is the same for bosons and fermions, it does not necessarily require the density of these states to be equal. In these circumstances, the Witten index is given by the sum of bulk and deficit contributions (see [22, 23] and references therein). The former is our partition function.

However, in one particular case, $N = 1$, we can calculate the bulk contribution to the index [24]. This case describes instantons in an abelian U(1) gauge theory which only become non trivial on a non-commutative background [19]. For $N = 1$, the undeformed ADHM constraints (1.4), are solved by $q = \tilde{q} = 0$ and x and $\tilde{x}$ diagonal; so $\mathfrak{M}_{k,1} = \mathrm{Sym}_k(\mathbb{R}^4)$. The symmetric product has orbifold singularities whenever two points come together. On turning on the FI couplings, the singularities are resolved and the space $\mathfrak{M}^{(\zeta)}_{k,1}$ is smooth. However, it is still non compact since the instantons can move apart in $\mathbb{R}^4$. The strategy [24] to calculate the bulk contribution to the $\mathcal{L}^2$-index, is to calculate the boundary contribution using a generalization of an argument due to Yi [22], and developed by Green and Gutperle [25], and then use the fact that the index is known to be 1.[3] In this way we find

$$\widehat{\mathcal{Z}}_{k,1}(\zeta, \infty) = \sum_{d|k} \frac{1}{d}\,, \tag{1.12}$$

where the sum is over the integer divisors of k.

Thinking of our partition function as the contribution to a Witten index is very useful because we normally can expect these kinds of quantities to be independent of deformations of the quantum mechanical system. However, since our space is non compact and the quantum mechanical potential has flat directions we have to be careful. In this case the bulk contribution to the Witten index need not be independent of a deformation which alters the long-range behaviour of the potential. However, suppose we turn on VEVs φ, i.e. separate the D4-branes. This corresponds to turning on a superpotential in the supersymmetric quantum mechanics. We will find that this does change the partition function, $\widehat{\mathcal{Z}}_{k,N}(\zeta,\infty,\varphi) \neq \widehat{\mathcal{Z}}_{k,N}(\zeta,\infty,0)$, but the resulting quantity is independent of the VEVs. Hence, we can take the VEVs large and evaluate the partition function by localization on the minima of the superpotential in the standard way [27]. Rather than think of this in terms of the quantum mechanics in one dimension higher, we can just as well consider localization at the level of the matrix integral. These techniques will give us a very powerful way to potentially evaluate the instanton partition function in certain circumstances; for

[3] This fact follows from the strong-weak coupling duality of the theory of a D4-brane in type-IIA string theory and the theory of an M5-brane in M-theory: see [26] and references therein.

$\widehat{\mathcal{Z}}_{1,N}(\zeta,g_0,\varphi)$	$\zeta=0=g_0^{-1}$	$\zeta=0, g_0^{-1}\neq 0$	$\zeta\neq 0, g_0^{-1}=0$	$\zeta\neq 0, g_0^{-1}\neq 0$
$\varphi=0$	0	$\frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)}$	$\frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)}$	$\frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)}$
$\varphi\neq 0$	$N-\frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)}$	N	N	N

Table 1: The values of $\widehat{\mathcal{Z}}_{1,N}(\zeta,g_0,\varphi)$.

example, in $\mathcal{N}=4$ SYM_4 on the the Coulomb branch with non-commutativity. In these cases, we shall reduce the problem to one involving abelian instantons (1.12).

When the FI couplings vanish the target space is no longer smooth. With non-vanishing VEVs, the partition function is still well defined. We will show by explicit calculation in the one-instanton sector that localization also occurs in this case. The partition function receives contributions from the same abelian instanton subspaces as before, but now there is an additional contribution from the small instanton singularity of the moduli space. At the present, we have not developed a way to calculate the contributions from the small instanton singularities for $k>1$.

Up till now, we have been considering the decoupling limit $g_0=\infty$ ($\alpha'=0$). However, there are some applications where we need to think about about genuine D-instantons rather than gauge theory instantons, and in these circumstances the g_0^{-2} stringy coupling terms in the action (1.1) become important. For example, Green and Gutperle [21] consider D-instanton effects in the effective action of a single D3-brane which depend on the partition function $\widehat{\mathcal{Z}}_{k,1}(0,g_0)$. We shall find by explicit calculations in the one-instanton sector that the string corrections have the same effect as non-commutativity in that they regularize the behaviour at the singularities in the instanton moduli space. There are strong indications that this generalizes to arbitrary instanton charge:

$$\widehat{\mathcal{Z}}_{k,N}(\zeta,g_0,\varphi)=\widehat{\mathcal{Z}}_{k,N}(\zeta,\infty,\varphi)=\widehat{\mathcal{Z}}_{k,N}(0,\infty,\varphi)\,. \tag{1.13}$$

2. The one-instanton sector

In this section, we consider the one-instanton sector $k=1$ where we can evaluate the D-instanton partition function by brute force. The results of this section for $k=1$ are summarized in the table 1. We will use these calculations to prime our intuition for the multi-instanton cases where such frontal assaults are not feasible.

2.1 Collective coordinates and singularities

The D-instanton matrix theory has an SU(4) symmetry which is the covering group of the Lorentz group in six dimensions: the maximal dimension in which the Dp/D$(p+4)$-brane system can be formulated. In four dimensions, the covering group of the Lorentz group is $\mathrm{SU}(2)_X\times\mathrm{SU}(2)_Y\subset\mathrm{SU}(4)$. Spinor indices of SU(4) are denoted

$A, B = 1, \ldots, 4$. A vector representation of $\mathrm{SO}(6) \simeq \mathrm{SU}(4)$ will be denoted χ_a, $a = 1, \ldots, 6$, or alternatively as an antisymmetric SU(4) representation $\chi_{AB} = -\chi_{BA}$ subject to the reality condition $(\chi^\dagger)^{AB} = \frac{1}{2}\epsilon^{ABCD}\chi_{CD}$. In the D$(-1)$/D3-brane system, the covering group of the Lorentz group of the D3-brane theory is $\mathrm{SU}(2)_L \times \mathrm{SU}(2)_R$, with the usual $\alpha, \dot\alpha = 1, 2$ indices, where the second factor is identified with a subgroup of the $\mathrm{SU}(2)_R \times \mathrm{U}(1)$ R-symmetry of the D-instanton theory.

In this section we will use the notation for the instanton calculus which is taken from [4]. In brief, the ADHM variables are related to the scalars in the hypermultiplets by[4]

$$w_{\dot\alpha} \equiv \begin{pmatrix} q^\dagger \\ \tilde q \end{pmatrix}, \qquad \bar w^{\dot\alpha} \equiv \begin{pmatrix} q & \tilde q^\dagger \end{pmatrix}, \qquad a'_{\alpha\dot\alpha} \equiv \begin{pmatrix} x^\dagger & \tilde x \\ -\tilde x^\dagger & x \end{pmatrix}. \tag{2.1}$$

The Grassmann collective coordinates $\{\mu^A, \bar\mu^A\}$ are the dimensional reduction of the fermions from the fundamental hypermultiplets. On dimensional reduction to zero dimensions, the vector multiplet consists of an SO(6) vector χ_a, or χ_{AB}, coming from the complex scalar field in Φ and the components of the four-dimensional gauge field. In addition, there are 3 variables D^c, $c = 1, 2, 3$, arising from the dimensional reduction of the auxiliary fields. Finally, the fermions of the vector multiplet are $\lambda^{\dot\alpha}_A$.

The instanton has a scale size $\rho^2 = \frac{1}{2}\bar w^{\dot\alpha} w_{\dot\alpha}$. The remaining $4N - 5$ collective coordinates describe how the instanton is embedded in the gauge group. This may be specified by the SU(2) subgroup of the gauge group $w_{\dot\alpha}(\tau^c)^{\dot\alpha}{}_{\dot\beta}\bar w^{\dot\beta}$, $c = 1, 2, 3$. To get a feel for the nature of the singularities of the instanton moduli space, consider the case of a single instanton in SU(2). In this case, the (centered) instanton moduli space is simply the orbifold

$$\widehat{\mathfrak{M}}_{1,2} = \frac{\mathbb{R}^4}{\mathbb{Z}_2}, \tag{2.2}$$

where ρ is the radius and the SU(2) gauge group is parameterized by the S^3 solid angle.[5] Now consider the resolved space $\widehat{\mathfrak{M}}^{(\zeta)}_{1,N}$. It is convenient to take, without-loss-of-generality, $\zeta_{\mathbb{C}} = 0$ and $\zeta_{\mathbb{R}} > 0$. In this case, for fixed $\tilde q$ the solution to $qq^\dagger = \zeta_{\mathbb{R}} + \tilde q^\dagger \tilde q$, modulo the U(1) gauge symmetry is topologically $\mathbb{CP}^{N-1}$. Given a point q on $\mathbb{CP}^{N-1}$, the complex equation $q\tilde q = 0$ simply says that $\tilde q$ is a cotangent vector. Hence the resolved moduli space $\widehat{\mathfrak{M}}^{(\zeta)}_{1,N}$ is topologically the cotangent bundle $T^*\mathbb{CP}^{N-1}$ [20]. In particular, we can now see that the resolution of the singularity involves a blow up

[4]In general, the indices $i, j, \ldots = 1, \ldots, k$ and $u, v, \ldots = 1, \ldots, N$, however, in the one-instanton sector the i, j-indices are not required and moreover the adjoint hypermultiplets are completely decoupled. To compare to previous works on the instanton calculus, here, we are taking a euclidean version of the σ-matrices: $\sigma^n_{\alpha\dot\alpha} = (-1, i\tau^c)$ and $\bar\sigma^{\dot\alpha\alpha}_n = (-1, -i\tau^c)$.

[5]The $\mathbb{Z}_2$ factor corresponds to the centre of the gauge group.

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on $\mathbb{CP}^{N-1}$. Notice that the scale size is given by

$$\rho^2 = \tilde{q}^\dagger \tilde{q} + \frac{1}{2}\zeta_{\mathbb{R}} \,, \tag{2.3}$$

and so the minimum value of ρ is given by $\sqrt{\zeta_{\mathbb{R}}/2}$.

2.2 The D-instanton partition function

We begin with the most general case with VEVs, FI and g_0^{-2} couplings. The properly normalized centered instanton partition function is derived in appendix A (see eq. (A.6)) based on the formulae of [4]. For one instanton we have

$$\begin{aligned} \widehat{\mathcal{Z}}_{1,N}(\zeta, g_0, \varphi) = {} & 2^{-2N-1}\pi^{-6N-9} \int d^{2N}w\, d^{2N}\bar{w}\, d^6\chi\, d^3D\, d^{4N}\mu\, d^{4N}\bar{\mu}\, d^8\lambda\, \times \\ & \times e^{-\bar{w}^{\dot\alpha}\tilde{\chi}^2 w_{\dot\alpha} - iD^c((\tau^c)^{\dot\alpha}{}_{\dot\beta}\bar{w}^{\dot\alpha}w_{\dot\beta} - \zeta^c) - 2g_0^{-2}D^2 + 2\sqrt{2}i\pi\bar{\mu}^A\tilde{\chi}_{AB}\mu^B + i\pi(\bar{\mu}^A w_{\dot\alpha} + \bar{w}_{\dot\alpha}\mu^A)\lambda^{\dot\alpha}_A} \,. \end{aligned} \tag{2.4}$$

In the above, the 6-vector quantity $\tilde{\chi}_a$ includes the coupling to the VEVs φ_{au}:

$$(\tilde{\chi}_a)_{ij,uv} = (\chi_a)_{ij}\delta_{uv} - \delta_{ij}\varphi_{au}\delta_{uv} \,, \tag{2.5}$$

(although in the one-instanton sector $k = 1$ and the i, j-indices are not required).

The $\{\mu^A, \bar{\mu}^A\}$ integrals can be done by completing the square of the fermionic terms:

$$\begin{aligned} & 2\sqrt{2}i\pi \left[\bar{\mu}^A + \frac{1}{2\sqrt{2}}\lambda_{\dot\alpha A}\bar{w}^{\dot\alpha}(\tilde{\chi}^{-1})^{AB}\right] \tilde{\chi}_{BC} \left[\mu^C + \frac{1}{2\sqrt{2}}(\tilde{\chi}^{-1})^{CD} w_{\dot\alpha}\lambda^{\dot\alpha}_D\right] - \\ & \quad - \frac{i\pi}{2\sqrt{2}}\lambda_{\dot\alpha A}\bar{w}^{\dot\alpha}(\tilde{\chi}^{-1})^{AB} w_{\dot\beta}\lambda^{\dot\beta}_B \,. \end{aligned} \tag{2.6}$$

So integrating $\{\mu^A, \bar{\mu}^A\}$ gives a determinant factor

$$2^{6N}\pi^{4N}\det{}_{4N}\tilde{\chi} = \pi^{4N}\prod_u \tilde{\chi}_u^4 \,, \tag{2.7}$$

where we have introduced the six-vector $\tilde{\chi}_u$, the diagonal components of $\tilde{\chi}$:

$$\tilde{\chi}_u = \chi - \varphi_u \,. \tag{2.8}$$

2.3 When the VEV vanish

The case when the VEVs vanish is much simpler because the $\mathrm{U}(N)$ flavour symmetry is then unbroken. In this case, we can change variables from $w_{\dot\alpha}$ to the $\mathrm{U}(N)$-invariant coordinates $\{W^0, W^c\}$ [4]:

$$W^0 = \bar{w}^{\dot\alpha}w_{\dot\alpha} \,, \qquad W^c = (\tau^c)^{\dot\alpha}{}_{\dot\beta}\bar{w}^{\dot\beta}w_{\dot\alpha} \,. \tag{2.9}$$

The jacobian for the change of variables involves the volume for the U(N) orbit [4]:

$$\int d^{2N}w\, d^{2N}\bar{w} = \frac{2\pi^{2N-1}}{\Gamma(N)\Gamma(N-1)} \int dW^0\, d^3W^c\, \left[(W^0)^2 - |W^c|^2\right]^{N-2} . \tag{2.10}$$

In addition, it is important to notice that the range of integration over W^0 is limited to $W^0 \geq |W^c|$.

In terms of these variables, our integral is

$$\begin{aligned}\widehat{\mathcal{Z}}_{1,N}(\zeta, g_0, 0) = {} & \frac{2^{-2N-6}\pi^{-10}}{\Gamma(N)\Gamma(N-1)} \int dW^0\, d^3W^c\, d^6\chi\, d^3D\, d^8\lambda\, \left[(W^0)^2 - |W^c|^2\right]^{N-2} \chi^{4N} \times \\ & \times e^{-W^0\chi^2 - iD^c(W^c-\zeta^c) - 2g_0^{-2}|D^c|^2 - \frac{i\pi}{2\sqrt{2}}W^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}(\chi^{-1})^{AB}\lambda_{\dot\alpha A}\lambda^{\dot\beta}_B} . \end{aligned} \tag{2.11}$$

The λ integrals give

$$\int d^8\lambda\, e^{-\frac{i\pi}{2\sqrt{2}}W^c(\tau^c)^{\dot\alpha}{}_{\dot\beta}(\chi^{-1})^{AB}\lambda_{\dot\alpha A}\lambda^{\dot\beta}_B} = \pi^4 \frac{|W^c|^4}{\chi^4} . \tag{2.12}$$

The χ integral is then straightforward:

$$\int d^6\chi\, \chi^{4(N-1)} e^{-W^0\chi^2} = \frac{\pi^3\Gamma(2N+1)}{2} (W^0)^{-2N-1} , \tag{2.13}$$

as is the W^0 integral:

$$\int_{|W^c|}^{\infty} dW^0\, \left[(W^0)^2 - |W^c|^2\right]^{N-2} (W^0)^{-2N-1} = \frac{1}{2N(N-1)|W^c|^4} . \tag{2.14}$$

The remaining integrals are

$$\int d^3W^c\, d^3D^c\, e^{-2g_0^{-2}D^cD^c - iD^c(W^c-\zeta^c)} . \tag{2.15}$$

With the g_0^{-2} coupling present, the integrals over D^c is gaussian and leaves, in turn, a gaussian integral over W^c:

$$\left(\frac{\pi g_0}{2}\right)^{3/2} \int d^3W^c\, e^{-g_0^2(W^c-\zeta^c)^2/8} = 2^3\pi^3 . \tag{2.16}$$

On the other hand if $g_0 = \infty$, then the integral over D^c yields δ-functions:

$$2^3\pi^3 \int d^3W^c\, \delta^{(3)}(W^c - \zeta^c) = 2^3\pi^3 . \tag{2.17}$$

Hence, our first result is that the partition function $\widehat{\mathcal{Z}}_{1,N}(\zeta, g_0, 0)$ is actually independent of the dimensionless combination $g_0\zeta$:

$$\widehat{\mathcal{Z}}_{1,N}(\zeta, g_0, 0) = \frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)} . \tag{2.18}$$

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In particular, we can legitimately take $g_0 = \infty$ or $\zeta = 0$:

$$\widehat{\mathfrak{Z}}_{1,N}(\zeta, \infty, 0) = \widehat{\mathfrak{Z}}_{1,N}(0, g_0, 0) = \frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)}, \tag{2.19}$$

as long as ζ, respectively g_0, are finite so that the integral does not vanish due to superconformal invariance. We remark that the left-hand side is precisely the GBC integral on $\widehat{\mathfrak{M}}^{(\zeta)}_{1,N}$. This follows from general principles [24]. The main point is that with $g_0 = \infty$, we can integrate out the gauge field χ_a via its equation-of-motion. This generates a four-fermion interaction involving the Riemann tensor of the resolved instanton moduli space and provides the usual Grassmann representation of the GBC integral.

Notice that the resolution of the singularities that is provided by non-commutativity, $\zeta \neq 0$, can apparently be traded for string g_0^{-2} couplings at the level of the partition function. This is very reminiscent of what happens in the DLCQ description of $\mathcal{N} = (2,0)$ "little string theory", where the rather complicated regularization of the singularities of the instanton moduli space provided by string theory, can be traded for non-commutativity [13, 14].

2.4 With non-zero VEVs

Now we consider the case where we turn on the VEVs.[6] Initially, we shall assume that there are no FI or g_0^{-2} couplings. The VEVs explicitly break the $\mathrm{U}(N)$ flavour symmetry and it is no longer possible to make the change of variables (2.9) and we must work directly with the $w_{\dot{\alpha}}$ variables. However, the exponential in the integrand is quadratic in the w's and so the integrals can be done explicitly using[7]

$$\int d^2 w_u \, d^2 \bar{w}_u \, e^{-A_0 \bar{w}_u^{\dot{\alpha}} w_{u\dot{\alpha}} + iA^c (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \bar{w}_u^{\dot{\beta}} w_{u\dot{\alpha}}} = \frac{4\pi^2}{(A^0)^2 + A^c A^c}. \tag{2.20}$$

So in our case we generate

$$(4\pi^2)^N \prod_u \frac{1}{\tilde{\chi}_u^4 + (D + \Xi_u)^2}, \tag{2.21}$$

where we have defined the 3-vectors Ξ_u with components

$$\Xi_u^c = \frac{\pi}{2\sqrt{2}} (\tau^c)^{\dot{\alpha}}{}_{\dot{\beta}} \lambda_{\dot{\alpha} A} (\tilde{\chi}_u^{-1})^{AB} \lambda_B^{\dot{\beta}}. \tag{2.22}$$

Notice that Ξ_u is quadratic in the remaining Grassmann variables $\lambda_A^{\dot{\alpha}}$.

[6] We will always assume that they are generic.

[7] We follow very closely the approach of [3] which considers a similar partition function in one the instanton sector of $\mathcal{N} = 2$ gauge theory.

The most arduous part of the calculation is now upon us: we must integrate out the λ's and unfortunately this has to done by brute force. To start with

$$\int d^8\lambda\, F = \frac{1}{4!}\sum_{u_1u_2u_3u_4}\int d^8\lambda\; \Xi_{u_1}^{c_1}\Xi_{u_2}^{c_2}\Xi_{u_3}^{c_3}\Xi_{u_4}^{c_4}\frac{\partial}{\partial\Xi_{u_1}^{c_1}}\frac{\partial}{\partial\Xi_{u_2}^{c_2}}\frac{\partial}{\partial\Xi_{u_3}^{c_3}}\frac{\partial}{\partial\Xi_{u_4}^{c_4}}F\Big|_{\Xi=0}, \tag{2.23}$$

where

$$F = \prod_u f_u\,, \qquad f_u = \frac{\tilde\chi_u^4}{\tilde\chi_u^4 + (D+\Xi_u)^2}\,. \tag{2.24}$$

The integrals over the λ's yield

$$\begin{aligned}\int d^8\lambda\, \Xi_{u_1}^{c_1}\Xi_{u_2}^{c_2}\Xi_{u_3}^{c_3}\Xi_{u_4}^{c_4} = 2^8\pi^4\frac{1}{\tilde\chi_{u_1}^2\cdots\tilde\chi_{u_4}^2}\Big(&\tilde\chi_{u_1}\cdot\tilde\chi_{u_3}\tilde\chi_{u_2}\cdot\tilde\chi_{u_4} + \tilde\chi_{u_1}\cdot\tilde\chi_{u_4}\tilde\chi_{u_2}\cdot\tilde\chi_{u_3} -\\ &- \tilde\chi_{u_1}\cdot\tilde\chi_{u_2}\tilde\chi_{u_3}\cdot\tilde\chi_{u_4}\Big)\delta^{c_1c_2}\delta^{c_3c_4} +\\ &+\text{permutations of }(1234)\,.\end{aligned} \tag{2.25}$$

Therefore (2.23) is equal to

$$\begin{aligned}2^8\pi^4\sum_{u_1u_2u_3u_3}\frac{1}{\tilde\chi_{u_1}^2\tilde\chi_{u_2}^2\tilde\chi_{u_3}^2\tilde\chi_{u_4}^2}\Big(&\tilde\chi_{u_1}\cdot\tilde\chi_{u_3}\tilde\chi_{u_2}\cdot\tilde\chi_{u_4} + \tilde\chi_{u_1}\cdot\tilde\chi_{u_4}\tilde\chi_{u_2}\cdot\tilde\chi_{u_3} -\\ &- \tilde\chi_{u_1}\cdot\tilde\chi_{u_2}\tilde\chi_{u_3}\cdot\tilde\chi_{u_4}\Big)\times\\ &\times\left(\frac{\partial}{\partial\Xi_{u_1}}\cdot\frac{\partial}{\partial\Xi_{u_2}}\right)\left(\frac{\partial}{\partial\Xi_{u_3}}\cdot\frac{\partial}{\partial\Xi_{u_4}}\right)F\Big|_{\Xi=0}.\end{aligned} \tag{2.26}$$

Notice that f_u is a function of $(D+\Xi_u)^2/(\chi-\varphi_u)^4$ alone; hence

$$\frac{(\chi-\varphi_u)^a}{(\chi-\varphi_u)^2}\frac{\partial F}{\partial\Xi_u^c} \equiv \frac{(D+\Xi_u)^c}{2(D+\Xi_u)^2}\frac{\partial F}{\partial\varphi_u^a}\,. \tag{2.27}$$

We can use this identity in (2.26) to trade Ξ_u-derivatives for φ_u-derivatives. One readily shows that

$$\int d^8\lambda\prod_u\frac{\tilde\chi_u^4}{\tilde\chi_u^4+(D+\Xi_u)^2} = \frac{16\pi^4}{D^4}\left(\sum_u\frac{\partial}{\partial\varphi_u}\cdot\sum_v\frac{\partial}{\partial\varphi_v}\right)^2\prod_u\frac{\tilde\chi_u^4}{\tilde\chi_u^4+D^2}\,. \tag{2.28}$$

Now we can trade the derivatives over the 6-vectors φ_u for those over the 6-vector χ:

$$\sum_u\frac{\partial}{\partial\varphi_u}\longrightarrow\frac{\partial}{\partial\chi}\equiv\nabla_\chi\,. \tag{2.29}$$

We are now in a position to integrate out the Lagrange multipliers of the ADHM constraints D^c. Writing

$$\widehat{\mathcal{Z}}_{1,N}(0,\infty,\varphi) = \int d^6\chi\,\left(\nabla_\chi\cdot\nabla_\chi\right)^2 I\,,\qquad I = \frac{1}{2^5\pi^5}\int\frac{d^3D}{D^4}\prod_u\frac{\tilde\chi_u^4}{\tilde\chi_u^4+D^2}\,. \tag{2.30}$$

The integrand is only a function of $\xi = |D|$, and so the angular integrals are trivial, leaving

$$I = \frac{1}{8\pi^4}\int_0^\infty d\xi \frac{1}{\xi^2}\prod_u \frac{\tilde{\chi}_u^4}{\tilde{\chi}_u^4+\xi^2}\,. \tag{2.31}$$

Since the integrand is an even function of ξ we can extend the range of integration from $(0,\infty)$ to $(-\infty,\infty)$ and evaluate it as a contour integral. Completing the contour from $\xi=-\infty$ to $\xi=+\infty$ by the semi-circle in the upper half plane, we pick up residues of the N simple poles at $\xi = i\tilde{\chi}_u^2$. The integral is singular due to the double pole on the real axis at $\xi=0$, however, the residue is independent of χ and so this singularity will not actually contribute to (2.30). Hence, up to this unimportant singularity, the result of the integral is

$$I = -\frac{1}{16\pi^3}\sum_u \frac{1}{\tilde{\chi}_u^2}\prod_{v\neq u}\frac{\tilde{\chi}_v^4}{\tilde{\chi}_v^4-\tilde{\chi}_u^4}\,. \tag{2.32}$$

It only remains for us to integrate over χ:

$$\widehat{\mathcal{Z}}_{1,N}(0,\infty,\varphi) = -\frac{1}{16\pi^3}\int d^6\chi \left(\nabla_\chi\cdot\nabla_\chi\right)^2\sum_u \frac{1}{\tilde{\chi}_u^2}\prod_{v\neq u}\frac{\tilde{\chi}_v^4}{\tilde{\chi}_v^4-\tilde{\chi}_u^4}\,. \tag{2.33}$$

Rather remarkably, however, the integral is a total derivative and we can evaluate it using Stokes' theorem. This fact is very significant because it means that the integral only picks up contributions from certain points on the moduli space. We will have more to say about this later.

The integral will potentially pick up contributions from any singularities of the integrand as well as from the sphere at infinity. Contrary to appearances the integrand is *not* singular at $\tilde{\chi}_u^2=\tilde{\chi}_v^2$ due to the cancellation between u-th and v-th terms in the sum. However, there are N singularities at $\tilde{\chi}_u=0$, i.e. $\chi=\varphi_u$, $u=1,\ldots,N$. In the vicinity of these singularities the integrand behaves as

$$-\frac{1}{16\pi^3|\chi-\varphi_u|^2}+\cdots\,. \tag{2.34}$$

The contribution to the integral can be evaluated by surrounding the point $\chi=\varphi_u$ by a small sphere of radius r. The contribution is then

$$-\operatorname{Vol}(S^5)\cdot\lim_{r\to 0} r^5\frac{d}{dr}r^{-5}\frac{d}{dr}r^5\frac{d}{dr}\left(-\frac{1}{16\pi^3 r^2}\right)=1\,. \tag{2.35}$$

Hence, each of the N singularities contributes $+1$ to the final answer. At this point we remark that these contributions come from the zeros of the potential that is induced in the matrix integral when the VEVs are turned on. Indeed from (2.4), we see that the potential is zero when

$$w_{u\dot{\alpha}}\chi_a - \varphi_{ua}w_{u\dot{\alpha}} = 0 \qquad \text{(no sum on } u)\,. \tag{2.36}$$

There are N solutions of these equations with $\chi = \varphi_u$, $u = 1, \ldots, N$, and $w_{v\dot\alpha} = 0$, $v \neq u$. This matches the positions of the singularities exactly. However, we could also have $w_{\dot\alpha} = 0$, which is precisely the small instanton singularity, and this gives a contribution that corresponds to the sphere at infinity in χ-space and which we evaluate below.

To complete the evaluation of the integral we have to consider the contribution from the large sphere at infinity. Consider the behaviour of the integrand as a function of $r = |\chi|$. Naïvely, it looks like

$$\sum_u \frac{1}{\tilde\chi_u^2} \prod_{v\neq u} \frac{\tilde\chi_v^4}{\tilde\chi_v^4 - \tilde\chi_u^4} \sim r^{N-2}\,, \tag{2.37}$$

for large r. This, if true, would be disastrous; however, just as there are no singularities at $\tilde\chi_u^4 = \tilde\chi_v^4$, it turns out that (2.37) is misguided. A more careful analysis shows

$$\lim_{r\to\infty} \sum_u \frac{1}{\tilde\chi_u^2} \prod_{v\neq u} \frac{\tilde\chi_v^4}{\tilde\chi_v^4 - \tilde\chi_u^4} = k_N r^{-2} + \mathcal{O}(r^{-4})\,, \tag{2.38}$$

where

$$k_N = \frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)}\,. \tag{2.39}$$

This gives the following boundary contribution to the integral from the sphere at finity:

$$\mathrm{Vol}\left(S^5\right) \cdot \lim_{r\to\infty} r^5 \frac{d}{dr} r^{-5} \frac{d}{dr} r^5 \frac{d}{dr}\left(-\frac{k_N}{16\pi^3 r^2}\right) = -k_N\,. \tag{2.40}$$

We remark at this point that this contribution can be thought of as coming from the small instanton singularity on the moduli space, as we alluded to above. The point is that if we had chosen to integrate out the χ variable first, rather than $w_{\dot\alpha}$, then

$$\chi_{AB} = \rho^{-2}\left(\bar w^{\dot\alpha}\varphi_{AB} w_{\dot\alpha} + \sqrt{2}i\pi\epsilon_{ABCD}\bar\mu^C\mu^D\right). \tag{2.41}$$

So large χ corresponds to small ρ.

Summing up the contributions, we have

$$\widehat{\mathcal{Z}}_{1,N}(0,\infty,\varphi) = N - \frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)}\,. \tag{2.42}$$

Remember that even though the result does not depend on the VEVs, we cannot take $\varphi = 0$ because the result is discontinuous. The reason is clear, when $\varphi = 0$ the N singularities all merge to $\chi = 0$ and the integrand I is then identically equal to $-k_N/(16\pi^2\chi^2)$. So what happens in this case is that the contribution from the sphere at infinity cancels the contribution from the origin and $\widehat{\mathcal{Z}}_{1,N}(0,\infty,0) =$

0, as expected due to the unsaturated superconformal Grassmann integrals. On comparison with (2.19), the result appears very suggestive: the contribution from the singularity appears to be minus $\widehat{\mathcal{Z}}_{1,N}(\zeta,\infty,0)$. This connection can be made more precise as we shall see below.

We now consider the calculation above but with the addition of the FI couplings; so we are calculating the integral $\widehat{\mathcal{Z}}_{1,N}(\zeta,\infty,\varphi)$. We follow the steps as above up to the D integral (2.30). We now have to include the FI coupling which involves dependence on the angular coordinates of D^c. The angular integrals yield

$$\int d(\cos\theta)\,d\phi\,e^{i\zeta^c D^c}=\frac{2\pi}{i|\zeta|\xi}\left(e^{i|\zeta|\xi}-e^{-i|\zeta|\xi}\right). \tag{2.43}$$

The integral over $\xi=|D|$ is then modified from (2.30) to

$$I=\frac{1}{16i\pi^4|\zeta|}\int_0^\infty d\xi\frac{1}{\xi^3}\left(e^{i|\zeta|\xi}-e^{-i|\zeta|\xi}\right)\prod_u\frac{\tilde\chi_u^4}{\tilde\chi_u^4+\xi^2}. \tag{2.44}$$

The integrand is symmetric in ξ and so, as before, we can extend the range from $(0,\infty)$ to $(-\infty,+\infty)$. We then split the integral into two terms whose integrands depend on $e^{\pm i|\zeta|\xi}$ and evaluate them as contour integrals by completing the contours at infinity in the upper, and lower, half planes, respectively. As before there are simple poles at $\xi=\pm i\tilde\chi_u^2$; however, now the double pole at $\xi=0$ does contribute. One finds, up to a χ-independent singularity,

$$I=\frac{1}{16\pi^3|\zeta|}\left(\sum_u\frac{1}{\tilde\chi_u^4}e^{-|\zeta|\tilde\chi_u^2}\prod_{v\neq u}\frac{\tilde\chi_v^4}{\tilde\chi_v^4-\tilde\chi_u^4}-\sum_u\frac{1}{\tilde\chi_u^4}\right). \tag{2.45}$$

The integrand looks similar to (2.32) but with the addition of the $e^{-|\zeta|\tilde\chi_u^2}$ terms. As before there are singularities at $\chi=\varphi_u$, $u=1,\ldots,N$, but the extra ζ-dependence does not affect their residue. The behaviour near $\chi=\varphi_u$ is

$$I=-\frac{1}{16\pi^3|\chi-\varphi_u|^2}+\cdots, \tag{2.46}$$

i.e. these singularities yield the *same* contribution as in the $\zeta=0$ case. In this case, however, there is no contribution from the sphere at infinity due to the exponential fall off of the $e^{-|\zeta|\tilde\chi_u^2}$ terms in (2.45). This is exactly what we would have expected: when the FI couplings are turned on the singularity of the instanton moduli space is resolved and the space becomes smooth. Hence, the contribution from the singularity disappear.

To summarize, we only have the contributions from the N singularities giving

$$\widehat{\mathcal{Z}}_{1,N}(\zeta,\infty,\varphi)=N. \tag{2.47}$$

We can easily also extract the result for $\widehat{\mathcal{Z}}_{1,N}(\zeta,\infty,0)$, which is *not* simply equal to the limit as $\varphi\to 0$ of $\mathcal{Z}_{1,N}(\zeta,\infty,\varphi)$. With the VEVs set to zero, the singularities of I all merge to $\chi=0$. The relevant behaviour near $\chi=0$ for the quantity I, with $\tilde{\chi}_u=0$, is

$$I=-\frac{k_N}{16\pi^3|\chi|^2}+\cdots\,, \tag{2.48}$$

where k_N is the same constant (2.39) that appeared in (2.38). The sphere at infinity does not contribute and therefore

$$\widehat{\mathcal{Z}}_{1,N}(\zeta,\infty,0)=\frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)}\,. \tag{2.49}$$

The fact that the contribution from $\chi=0$ is the opposite of the contribution from the sphere at infinity in (2.42), as noted above, is because when $\zeta=0$ the contributions from $\chi=0$ and $|\chi|=\infty$ must precisely cancel because $\widehat{\mathcal{Z}}_{1,N}(0,\infty,0)$ vanishes due to unsaturated Grassmann integrals. To sum up, we can say

$$\mathcal{Z}_{1,N}(0,\infty,\varphi)=\mathcal{Z}_{1,N}(\zeta,\infty,\varphi)+\mathcal{Z}_{1,N}(\mathrm{sing})\,, \tag{2.50}$$

where

$$\mathcal{Z}_{1,N}(\mathrm{sing})=-\mathcal{Z}_{1,N}(\zeta,\infty,0)\,. \tag{2.51}$$

Finally, we consider the g_0^{-2} corrections. First of all, with $\zeta^c=0$, we have the modified integral (2.31) over $\xi=|D|$:

$$I=\frac{1}{8\pi^4}\int_0^\infty d\xi\,\frac{1}{\xi^2}\prod_u\frac{\tilde{\chi}_u^4}{\tilde{\chi}_u^4+\xi^2}e^{-2g_0^{-2}\xi^2}\,. \tag{2.52}$$

Up to the χ-independent singularity, this is equal to

$$I=-\frac{1}{16\pi^3}\sum_u\frac{1}{\tilde{\chi}_u^2}\left(1-\mathrm{erf}(2g_0^{-2}\tilde{\chi}_u^2)\right)e^{2g_0^{-2}\tilde{\chi}_u^2}\prod_{v\neq u}\frac{\tilde{\chi}_v^4}{\tilde{\chi}_v^4-\tilde{\chi}_u^4}\,, \tag{2.53}$$

where we have introduced the *error function* $\mathrm{erf}(z)=(2/\sqrt{\pi})\int_0^z e^{-x^2}dx$. The addition of the string coupling term does not alter the behaviour near the singularity $\chi=\varphi_u$ (2.34); hence, each of the N singularities contributes $+1$, as before. For large $r=|\chi|$, on the other hand,

$$\left(1-\mathrm{erf}(2g_0^{-2}\tilde{\chi}_u^2)\right)e^{2g_0^{-2}\tilde{\chi}_u^2}=\frac{g_0}{\sqrt{2\pi}r}+\mathcal{O}(r^{-2})\,, \tag{2.54}$$

and so there is no contribution from the sphere at infinity. Consequently

$$\widehat{\mathcal{Z}}_{1,N}(0,g_0,\varphi)=N\,. \tag{2.55}$$

It is easy to see that adding the FI coupling has no additional effect and therefore

$$\widehat{\mathfrak{Z}}_{1,N}(\zeta, g_0, \varphi) = N\,. \tag{2.56}$$

Just as in the case without VEVs, the string coupling has the same effect as the FI couplings. Now we see very explicitly that they both kill the contribution to the partition function from the small instanton singularity.

2.5 Lessons from the one-instanton sector

Before moving on, let us sum up what we have learnt from explicit calculation in the one-instanton sector.

(i) The quantity

$$\widehat{\mathfrak{Z}}_{1,N}(\zeta,\infty,0) = \frac{2\Gamma(N+1/2)}{\Gamma(1/2)\Gamma(N)}\,, \tag{2.57}$$

as mentioned previsouly, is the GBC integral of the resolved instanton moduli space $\widehat{\mathfrak{M}}^{(\zeta)}_{1,N}$. We can check this in the cases $N=1$ and 2. When $N=1$, the moduli space is simply a point and we have $\widehat{\mathfrak{Z}}_{1,1}(\zeta,\infty)=1$ in agreement with (1.12). When $N=2$ the unresolved space is $\widehat{\mathfrak{M}}_{1,2}=\mathbb{R}^4/\mathbb{Z}_2$. This is the same as the 2-instanton abelian instanton moduli space $\widehat{\mathfrak{M}}_{2,1}$. When we turn on the FI coupling and smooth out the singularity, the latter becomes the Eguchi-Hanson manifold [28]. The same resolution occurs for $\widehat{\mathfrak{M}}_{1,2}$ and consequently the GBC integral is the same for both. The GBC integral for the Eguchi-Hanson space is calculated in [29] to be 3/2 and so

$$\widehat{\mathfrak{Z}}_{1,2}(\zeta,\infty,0) \equiv \widehat{\mathfrak{Z}}_{2,1}(\zeta,\infty) = \frac{3}{2}\,, \tag{2.58}$$

which is in agreement with (2.57) and (1.12).

(ii) The partition function seems to enjoy some localization properties. When the FI couplings are non trivial, the partition function is independent of g_0 and is equal to the GBC integral of the resolved instanton moduli space: there is a localization on this moduli space. When the FI coupling vanishes, the partition function receives an additional contribution which can be thought of as coming from the small instanton singularity; however, this contribution is purely additive. This suggests that the two contributions are associated to the two branches of minima of the matrix theory action. The first branch is what we would call the Higgs branch in higher dimensions, since $\chi_a=0$ only $w_{\dot\alpha}$ are non vanishing, and this gives rise to the GBC integral over the resolved instanton moduli space. The second branch is the Coulomb branch on which $w_{\dot\alpha}$ vanishes but χ_a is non trivial. Moreover, we have shown that the contribution form the Coulomb branch can be related to that of the Higgs branch: they are equal and opposite.

(iii) When VEVs are turned on apparently there are additional localizations. The reason is that the matrix theory action now includes a VEV-dependent potential

$$V = \sum_{u=1}^{N} |(\chi_a - \varphi_{au}) w_{u\dot\alpha}|^2 . \tag{2.59}$$

When the FI couplings are non trivial, $w_{\dot\alpha}$ cannot vanish and so the only zeros of the potential are at $\chi_a = \varphi_{au}$, $u = 1, \ldots, N$, and $w_{\dot\alpha v} = 0$, for $v \neq u$. There are N solutions of this form corresponding to the choice of u. We found that $\widehat{\mathfrak{Z}}_{1,N}(\zeta, \infty, \varphi) = N$ which we shall argue in section 5 has the topological interpretation as the Euler characteristic of the resolved moduli space $\widehat{\mathfrak{M}}^{(\zeta)}_{1,N}$. When, additionally, the FI couplings vanish, the potential also vanishes when $w_{u\dot\alpha} = 0$, corresponding to the Coulomb branch contribution which is VEV independent.

The result described above is entirely consistent with the results of Lee and Yi [30] who considered instanton solitons in non-commutative SYM_5 compactified on a sphere in the decompactification limit. In this limit the one instanton moduli space $\mathfrak{M}^{(\zeta)}_{1,N}$ has the Calabi metric. Lee and Yi then identified precisely N ground-states of the instanton moduli space quantum mechanics when the gauge theory is on the coulomb branch matching our explicit evaluation of the D-instanton partition function.

(iv) The effect of turning on the string g_0^{-2} coupling is completely equivalent to having non-trivial FI couplings. We can see precisely why this so from (2.16) and (2.17). With $\zeta \neq 0$ and $g_0 = \infty$, the instanton size is prevented from going to zero by the modification of the ADHM constraints (1.11). On the contrary, with $\zeta = 0$ but g_0 finite, the momentum maps on the left-hand sides of (1.4) are no longer imposed as constraints; rather they are smeared over a scale g_0^{-1}. Either way has the effect of smoothing over the small instanton singularity and suppressing the contribution from the Coulomb branch.

3. The D-instanton partition function at large-N arbitrary k

In this section we show how the D-instanton partition function can be evaluated in the large-N limit for all instanton numbers. In fact, the necessary formalism has already been developed in [4], where the D-instanton partition function in the decoupling limit $g_0 = \infty$ and with no FI coupling was evaluated. In that case, since superconformal invariance was not broken, what was actually calculated was the partition function with the scale size and 8 Grassmann superconformal integrals, as well as the usual COM integrals, factored out. We will restrict our discussion to simply show how to modify the calculation of [4] to include non-trivial FI couplings.[8] We will only consider the decoupled case $g_0 = \infty$ here.

[8] We will draw extensively on [4] and use the notation there (which agrees with much of section 2) without explanation.

The large-N limit of the instanton partition function is tractable because there is a saddle-point approximation that captures the leading order behaviour in $1/\sqrt{N}$. From [4], the saddle-point equations for $\{\chi_a, W, a'_n\}$ are

$$\epsilon^{ABCD}\,(\mathbf{L}\cdot\chi_{AB})\,\chi_{CE} = \frac{1}{2}\delta^D_E\,1_{[k]\times[k]}\,, \tag{3.1a}$$

$$\chi_a\chi_a = \frac{1}{2}(W^{-1})^0\,, \tag{3.1b}$$

$$[\chi_a,[\chi_a,a'_n]] = i\bar\eta^c_{nm}[a'_m,(W^{-1})^c]\,. \tag{3.1c}$$

The maximally degenerate solution of these equations around which one develops a fluctuation analysis to capture the leading order behaviour in $1/\sqrt{N}$, is

$$W^0 = 2\rho^2\,1_{[k]\times[k]}\,,\qquad \chi_a = \rho^{-1}\hat\Omega_a\,1_{[k]\times[k]}\,,\qquad a'_n = -X_n\,1_{[k]\times[k]}\,. \tag{3.2}$$

Now we must consider how the presence of the FI coupling modifies the situation. Let us plug the ansatz (3.2) into the eqs. (3.1a)–(3.1c), but now with the FI parameter included. In this case, (3.1a) and (3.1c) are satisfied, however, (3.1b) becomes

$$\rho^{-2} = \frac{\rho^2}{\rho^4-\frac{1}{4}|\zeta^c|^2}\,. \tag{3.3}$$

Hence, in the presence of a non-trivial FI coupling, the solution of the saddle-point equations must have $\rho=\infty$. As in [4], we now have to consider fluctuation analysis around the saddle-point solution: the new ingredient being the fact that there will be an addition term in the expansion of the action which accounts for the fluctuation of ρ about ∞. The problem actually decouples into a piece describing the fluctuations of the $\mathrm{SU}(k)$-valued variables, which is identical to that in [4], and a piece describing the fluctuations of the $\mathrm{U}(1)$ components. The former leads to the partition function of $\mathcal{N}=1$ SYM_{10} dimensionally reduced to zero dimensions, while the latter is a one-instanton-type integral which we will do below.

The $\mathrm{U}(1)$ piece of the integral is precisely the integral that we performed in section 2.3 (with $g_0=\infty$) with N large.[9] The one-instanton integral that remains after integrating out all the Grassmann collective coordinates is (up to a normalization factor)

$$I_N = |\zeta^c|^4\int d^6\chi\,dW^0\,\left[(W^0)^2-|\zeta^c|^2\right]^{N-2}\chi^{4(N-1)}e^{-W^0\chi^2}\,. \tag{3.4}$$

[9]In [4] we actually resolved the Grassmann analogues of the ADHM constraints rather than left them in with Grassmann Lagrange multipliers $\lambda^{\dot\alpha}_A$. Both viewpoints are equivalent and in the former, respectively latter, approach the 8 superconformal modes are associated to the 8 Grassmann variables $\bar\eta^{\dot\alpha A}$, respectively $\lambda^{\dot\alpha}_A$.

In the large-N limit we make a rescaling $\chi \to \sqrt{N}\chi$, and so the saddle-point action is, with $W^0 = 2\rho^2$ and $r = |\chi|$,[10]

$$S = N\left(\log\left(\rho^4 - \frac{1}{4}|\zeta^c|^2\right) + 2\log r^2 - 2\rho^2 r^2\right). \tag{3.5}$$

As expected from the analysis of the full saddle-point equations above, the saddle-point is at $r = 0$ and $\rho = \infty$. Expanding around the saddle-point in the usual way we have, at leading order in he large-N limit,

$$I_N \underset{N\to\infty}{=} 2^{2N-1}\pi^{7/2}N^{2N-3/2}e^{-2N+1}\,. \tag{3.6}$$

Of course, we could equally as well have taken the large-N limit of the exact result in section 2.2.

Putting this together with the SU(k) partition function and taking careful account of all the numerical factors, we find

$$\widehat{\mathcal{Z}}_{k,N}(\zeta,\infty,0) \underset{N\to\infty}{=} \frac{\pi^{6k-13/2}\sqrt{N}}{2^{2k-3}}k^{3/2}\sum_{d|k} d^{-2}\,. \tag{3.7}$$

For $k = 1$, one can verify that this is consistent with the large-N limit of (2.19).

4. Localization of the partition function

As we have seen in section 2 by explicit calculation, the instanton partition function seems to localize around the zeros of the matrix theory action and this strongly suggests that some cohomological (topological) field theory considerations are at work in the D-instanton matrix theory. In this section, we develop this theme following closely the approach of [31] (see also [32] for a general discussion of the formalism and other references). The former reference considered the case of D-instantons in flat ten-dimensional space with no D3-branes present. In the present paper, we are considering the same system but with the addition of D3-branes, so the technical difference is the inclusion of the fundamental hypermultiplets. The formalism that we develop should be related to the approach developed in [33] for describing the $\mathcal{N} = 2$ instanton calculus in the language of cohomological field theory, but we shall not pursue this relationship here.

It is not that difficult to directly modify the formalism of [31] to incorporate the fundamental hypermultiplets; however, we will proceed more generally to start with. As usual the key to applying cohomological field theory ideas is to find a nilpotent fermionic symmetry $\mathcal{Q}$. It turns out that in our matrix theory there is considerable freedom in doing this. For instance, if we think of our theory as the dimensional

[10] Recall that $W^0 \geq |W^c|$ and so $\rho \geq 2^{-1/2}|\zeta^c|$.

reduction of a $\mathcal{N}=(4,4)$ theory in two dimensions then certain combinations of the supersymmetry charges will do as the generator of the nilpotent fermionic symmetry [34]. The D-terms, $\int d^4\theta\cdots$ are then $\mathcal{Q}$-exact, and standard arguments suggest that the partition function will be independent of the couplings to these terms. For instance, by realizing the $\mathcal{N}=1$ vector multiplet as a twisted chiral multiplet in two dimensions, the kinetic term of the vector multiplet in (1.1) can be written as a D-term and so, on dimensional reduction to zero dimensions, the D-instanton partition function should be independent of g_0. This is consistent with what we found in the one-instanton sector. More generally, we would like to argue that when an FI coupling is present, we can actually take $g_0=\infty$ and completely decouple the kinetic term.

Another way to define the nilpotent fermionic symmetry is to start with the D3/D7-brane system where the description of the D3-brane is a four-dimensional $\mathcal{N}=2$ theory with an adjoint and N fundamental hypermultiplets. On this theory we then perform the "topological twisting" procedure of Witten [35]. The four-dimensional theory has a Lorentz group that we previously called $\mathrm{SU}(2)_X\times\mathrm{SU}(2)_Y$ as well as an R-symmetry $\mathrm{SU}(2)_R\times\mathrm{U}(1)$. The 8 supercharges transform as a $(\mathbf{2},\mathbf{1},\mathbf{2})+(\mathbf{1},\mathbf{2},\mathbf{2})$ of $\mathrm{SU}(2)_X\times\mathrm{SU}(2)_Y\times\mathrm{SU}(2)_R$. Twisting implies defining a new Lorentz group $\mathrm{SU}(2)_X\times\mathrm{SU}(2)'_Y$, where $\mathrm{SU}(2)'_Y$ is the diagonal subgroup of $\mathrm{SU}(2)_Y\times\mathrm{SU}(2)_R$. There is precisely one supercharge $\mathcal{Q}$ which is a singlet of the new Lorentz group. It is also nilpotent $\mathcal{Q}^2=0$ (up to $\mathrm{U}(k)$ gauge transformations). From the point-of-view of the D-instanton matrix theory, we are simply picking out a distinguished supersymmetry transformation. The adjoint-valued fields of our theory are those of an $\mathcal{N}=4$ theory in four dimensions reduced to zero dimensions, and so is related to [31, 32]. The novel feature here is the existence of the fundamental hypermultiplets.

We will define the complex combinations, $y=A_1+iA_2$ and $\tilde{y}=A_3+iA_4$, of the four-dimensional gauge field. The vector multiplet also includes the complex scalar field Φ. We will think of A_μ and ϕ as forming an SO(6) vector χ_a. The remaining bosonic variables $\{x,\tilde{x}\}$ and $\{q,\tilde{q}\}$, which are all complex, come from the hypermultiplets. We denote the 6 complex bosonic variables $B_l\subset\{y,\tilde{y},x,\tilde{x},q,\tilde{q}^\dagger\}$, $l=1,\ldots,6$. In addition, there are eleven auxiliary fields. In the vector multiplet there is one real and one complex one, $H_{\mathbb{R}}$ and $H_{\mathbb{C}}$, respectively, and two complex ones from each of the hypermultiplets, $H^{(a)}_{\dot\alpha}$ and $H^{(f)}_{\dot\alpha}$, for the adjoint and fundamental hypermultiplets, respectively.[11] Here $\dot\alpha=1,2$, the R-symmetry index, labels the two complex auxiliary fields in each multiplet. We will think of the auxiliary fields as forming a large eleven-dimensional vector $\vec{H}$.

The fermionic variables are split into 3 sets. Firstly, each of the complex variables B_l, $l=1,\ldots,6$, has a superpartner Ψ_l under $\mathcal{Q}$:

$$\mathcal{Q}B_l=\Psi_l\,,\qquad \mathcal{Q}\Psi_l=\phi\cdot B_l\,. \tag{4.1}$$

[11] We denote $\bar{H}^{\dot\alpha}\equiv(H_{\dot\alpha})^\dagger$.

Here, $\phi \cdot B_l \equiv [\phi, B_l]$, if $B_l \subset \{y, \tilde{y}, x, \tilde{x}\}$, or $\phi \cdot B_l \equiv \phi B_l$, if $B_l \subset \{q, \tilde{q}^\dagger\}$. The second set of fermions $\vec{\chi}$ (not to be confused with χ_a) form a $\mathcal{Q}$-multiplet with the auxiliary fields:

$$\mathcal{Q}\vec{\chi} = \vec{H}\,, \qquad \mathcal{Q}\vec{H} = \phi \cdot \vec{\chi}\,. \tag{4.2}$$

The final fermionic variable is the superpartner of the conjugate of ϕ, which we denote $\bar{\phi}$:

$$\mathcal{Q}\bar{\phi} = \eta\,, \qquad \mathcal{Q}\eta = [\phi, \bar{\phi}]\,. \tag{4.3}$$

To complete the $\mathcal{Q}$-multiplet structure, ϕ is a singlet:

$$\mathcal{Q}\phi = 0\,. \tag{4.4}$$

From (4.1)–(4.4), we see that $\mathcal{Q}$ is nilpotent up to a gauge transformation generated by ϕ.

In order to define the action of the instanton matrix theory, we define the "equations" $\vec{\mathcal{E}}$ which are associated to each of the auxiliary fields:

$$\begin{aligned}
\mathcal{E}_{\mathbb{R}} &= g_0 q q^\dagger - g_0 \tilde{q}^\dagger \tilde{q} + g_0[x, x^\dagger] + g_0[\tilde{x}, \tilde{x}^\dagger] + g_0^{-1}[y, y^\dagger] + g_0^{-1}[\tilde{y}, \tilde{y}^\dagger]\,,\\
\mathcal{E}_{\mathbb{C}} &= g_0 q\tilde{q} + g_0[x, \tilde{x}] + g_0^{-1}[y^\dagger, \tilde{y}^\dagger]\,,\\
\mathcal{E}_1^{(a)} &= [x, y] - [\tilde{x}^\dagger, \tilde{y}^\dagger]\,, \qquad \mathcal{E}_2^{(a)} = [x, \tilde{y}] + [\tilde{x}^\dagger, y^\dagger]\,,\\
\mathcal{E}_1^{(f)} &= yq - \tilde{y}^\dagger \tilde{q}^\dagger\,, \qquad \mathcal{E}_2^{(f)} = \tilde{y} q + y^\dagger \tilde{q}^\dagger\,.
\end{aligned} \tag{4.5}$$

Given these definitions, we can now write the action of the theory as the $\mathcal{Q}$-exact expression

$$S = \frac{1}{\lambda}\mathcal{Q}\ \mathrm{tr}_k \left(\frac{1}{4}\eta[\phi, \bar{\phi}] + \vec{H}\cdot\vec{\chi} - i\vec{\mathcal{E}}\cdot\vec{\chi} - \frac{1}{2}\sum_{l=1}^{6}\left(\Psi_l^\dagger \bar{\phi}\cdot B_l + \Psi_l \bar{\phi}\cdot B_l^\dagger\right)\right). \tag{4.6}$$

In the above, we have introduced the inner-product

$$\vec{A}\cdot\vec{B} = \frac{1}{4}A_{\mathbb{R}}B_{\mathbb{R}} + A_{\mathbb{C}}B_{\mathbb{C}}^\dagger + A_{\mathbb{C}}^\dagger B_{\mathbb{C}} + \sum_{h=a,f}\left(\bar{A}^{(h)\dot{\alpha}}B_{\dot{\alpha}}^{(h)} + \bar{B}^{(h)\dot{\alpha}}A_{\dot{\alpha}}^{(h)}\right). \tag{4.7}$$

We have also introduced an auxiliary coupling constant λ which is set to 1 to reproduce the matrix theory action. When the fundamental hypermultiplets are absent then, up to the fact that the vector multiplet variables have a U(1) component, the theory reduces to the matrix theory arising from the dimensional reduction of $\mathcal{N} = 1$ SYM_{10} considered in [31]. This latter theory gives $\mathcal{N} = 4$ SYM_4 when dimensionally reduced, and so when the fundamental hypermultiplets are absent our formalism can be derived by dimensionally reduced the treatment of $\mathcal{N} = 4$ SYM_4 in [32].

We now want to incorporate both the FI coupling and VEVs into this cohomological description of the instanton matrix theory. The FI coupling simply corresponds to modifying the equations in the following way:

$$\mathcal{E}_{\mathbb{R}} \longrightarrow \mathcal{E}_{\mathbb{R}} - g_0 \zeta_{\mathbb{R}} 1_{[k]\times[k]}\,, \qquad \mathcal{E}_{\mathbb{C}} \longrightarrow \mathcal{E}_{\mathbb{C}} - g_0 \zeta_{\mathbb{C}} 1_{[k]\times[k]}\,. \tag{4.8}$$

This deforms the action by a $\mathcal{Q}$-exact term

$$\delta S = \frac{ig_0}{\lambda} \mathcal{Q}\ \mathrm{tr}_k \left(\frac{1}{4} \zeta_{\mathbb{R}} \chi_{\mathbb{R}} + \zeta_{\mathbb{C}} \chi_{\mathbb{C}}^\dagger + \zeta_{\mathbb{C}}^\dagger \chi_{\mathbb{C}} \right). \tag{4.9}$$

The VEVs are incorporated in the following way. First of all, the VEVs φ_a are associated with the scalar fields of the vector multiplet in the following way:

$$\phi \longleftrightarrow \varphi_1 + i\varphi_2\,, \qquad y \longleftrightarrow \varphi_3 + i\varphi_4\,, \qquad \tilde{y} \longleftrightarrow \varphi_5 + i\varphi_6\,. \tag{4.10}$$

The φ_a, $a = 3, 4, 5, 6$, components of the VEV couple by modifying the equations associated to the hypermultiplets:

$$\begin{aligned} \mathcal{E}_1^{(f)} &\longrightarrow \mathcal{E}_1^{(f)} + q(\varphi_3 + i\varphi_4) - \tilde{q}^\dagger(\varphi_5 - i\varphi_6)\,, \\ \mathcal{E}_2^{(f)} &\longrightarrow \mathcal{E}_2^{(f)} + q(\varphi_5 + i\varphi_6) + \tilde{q}^\dagger(\varphi_3 - i\varphi_4)\,. \end{aligned} \tag{4.11}$$

Finally, to incorporate the remaining components of the VEVs φ_a, $a = 1, 2$, we have to deform the $\mathcal{Q}$-action itself:

$$\begin{aligned} \mathcal{Q}B_l &= \Psi_l\,, \qquad \mathcal{Q}\Psi_l = \phi \cdot B_l + T_{\varphi_1 + i\varphi_2} \cdot B_l\,, \\ \mathcal{Q}\vec{\chi} &= \vec{H}\,, \qquad \mathcal{Q}\vec{H} = \phi \cdot \vec{\chi} + T_{\varphi_1 + i\varphi_2} \cdot \vec{\chi}\,. \end{aligned} \tag{4.12}$$

Here, T_ϵ is the action of the Lie algebra of the $U(1)^N$ subgroup of the U(N) flavour symmetry, with generator ϵ, on the variables B_l and equations $\vec{\mathcal{E}}$. In addition, in (4.6) we must shift $\bar{\phi} \to \bar{\phi} + T_{\varphi_1 - i\varphi_2}$.

Having succeeded in interpreting the D-instanton matrix theory in this cohomological field theory way, we can proceed to reap the benefits. First of all, since the action is $\mathcal{Q}$-exact would normally imply that the partition function is invariant under deformations. However, in our case the situation is more delicate since the underlying space is non compact. Consequently, we will have to quite careful in applying the usual arguments. Nevertheless, the partition function should be independent of ζ, g_0, φ, as well as the auxiliary coupling λ, as long as we are careful in taking a parameter to zero or infinity. This is completely consistent with our one-instanton result of the last section where we found that $\widehat{\mathcal{Z}}_{1,N}(\zeta, g_0, \varphi)$ was a constant with discontinuities when some of the quantities went to zero or infinity.

However, we can do much more with this formalism. Since the whole action is $\mathcal{Q}$-exact, we can evaluate it in the limit of small auxiliary coupling λ. In this limit, the partition function localizes around the zeros of the action and the fluctuations can be integrated out to leading order. The resulting expression should then be exact. The zeros of the action are at

$$[\chi_a,\chi_b]=[\chi_a,x]=[\chi_a,\tilde{x}]=\chi_a q-q\varphi_a=\tilde{q}\chi_a-\varphi_a\tilde{q}=0\,, \tag{4.13}$$

along with the modified ADHM constraints (1.11). These equations are precisely the equations that govern the classical phases structure the D-instanton theory. Hence, we can use the language of phases to describe the various contributions to the partition function.

If the VEVs are zero and ζ is non vanishing, then the solution to (4.13) must have $\chi_a=0$. This corresponds to localizing on the Higgs branch of the gauge theory. The three auxiliary fields from the vector multiplet are integrated out at gaussian order leaving a factor

$$\exp\left[-\frac{g_0^2}{\lambda^2}\,\mathrm{tr}_k\left(\frac{1}{4}\left(\mu_{\mathbb{R}}-\zeta_{\mathbb{R}}1_{[k]\times[k]}\right)^2+\left|\mu_{\mathbb{C}}-\zeta_{\mathbb{C}}1_{[k]\times[k]}\right|^2\right)\right], \tag{4.14}$$

where $\mu_{\mathbb{R}}$ and $\mu_{\mathbb{C}}$ are the moment maps of the ADHM hyper-Kähler quotient (the left-hand sides of (1.4)). To leading order, integrating over the fluctuations of $\{q,\tilde{q},x,\tilde{x}\}$ orthogonal to the ADHM moduli space then amounts to imposing the ADHM constraints via explicit δ-functions:

$$\int dq\,d\tilde{q}\,dx\,d\tilde{x}\;\delta(\mu_{\mathbb{R}}-\zeta_{\mathbb{R}}1_{[k]\times[k]})\delta(\mu_{\mathbb{C}}-\zeta_{\mathbb{C}}1_{[k]\times[k]})\,. \tag{4.15}$$

At leading order, the variables χ_a and their Grassmann partners $\lambda^{\dot{\alpha}}_A$ are integrated out through their coupling to the hypermultiplets, rather than through their kinetic terms. The latter produce Grassmann δ-functions for the fermionic analogues of the ADHM constraints. The former are integrated out at gaussian order through the coupling

$$\mathrm{tr}_k(\chi_a\mathbf{L}\chi_a)\,. \tag{4.16}$$

Here, $\mathbf{L}$ is the operator on $k\times k$ matrices, that plays a ubiquitous rôle in the instanton calculus [4]. In the present notation,

$$\mathbf{L}\cdot\Omega=\{qq^\dagger+\tilde{q}^\dagger\tilde{q},\Omega\}+[x,[x^\dagger,\Omega]]+[x^\dagger,[x,\Omega]]+[\tilde{x},[\tilde{x}^\dagger,\Omega]]+[\tilde{x}^\dagger,[\tilde{x},\Omega]]\,. \tag{4.17}$$

The quantity $(\det_{k^2}\mathbf{L})^{1/2}$ measures the volume of the $\mathrm{U}(k)$-orbit through a point on the moduli space and consequently the gaussian coupling for χ_a is non degenerate since we have resolved the small instanton singularities by having a non-trivial FI coupling. Integrating out χ_a produces a factor $(\det_{k^2}\mathbf{L})^{-3/2}$. What remains is precisely the volume form on the deformed moduli instanton moduli space $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$, that is the trivial generalization, to include the FI couplings in the ADHM constraints, of that constructed for instantons in $\mathcal{N}=4$ SYM_4 [4]. This is also the GBC integral of $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$. Summing up, we have shown

$$\widehat{\mathfrak{Z}}_{k,N}(\zeta,g_0,0)=\widehat{\mathfrak{Z}}_{k,N}(\zeta,\infty,0)=\int_{\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}} e\left(T^*\mathfrak{M}\right), \tag{4.18}$$

where $e(T^*\mathfrak{M})$ is the Euler density. Notice that the resulting expression is independent of g_0 as we earlier anticipated.

When the VEVs are non vanishing there are additional localizations on the moduli space. We will always assume that the VEVs, if non vanishing, are generic. The point is that with a non-trivial FI coupling $(qq^\dagger)_{ii}$ and $(\tilde{q}^\dagger\tilde{q})_{ii}$ must be non vanishing. Hence, the solution of (4.13) requires that for a given i, there is only one value of $u_i\in\{1,\ldots,N\}$ for which q_{iu} and $\tilde{q}_{ui}$ are non vanishing. Then

$$(\chi_a)_{ij}=\varphi_{au_i}\delta_{ij}\,. \tag{4.19}$$

The equations $[\chi_a,x]=[\chi_a,\tilde{x}]=0$, imply that x_{ij} and $\tilde{x}_{ij}$ are only non vanishing if $u_i=u_j$. The ADHM constraints (1.11) then decouple in blocks associated to the set of i's with the same value of u_i. So, up to the $\mathrm{U}(k)$ symmetry we can associate a solution to a partition $k\to\{k_1,\ldots,k_p\}$. Since in a given block there is only one value of u, namely u_i, for which q_{iu} and $\tilde{q}_{ui}$ are non vanishing, the remaining ADHM constraints in that block are precisely those of an abelian instanton theory $N=1$. The solution space associated to the partition is precisely a product of abelian instanton spaces:

$$\frac{\mathfrak{M}^{(\zeta)}_{k_1,1}\times\cdots\times\mathfrak{M}^{(\zeta)}_{k_p,1}}{\mathbb{R}^4}\,. \tag{4.20}$$

This a rather nice interpretation in terms of D-instantons and D3-branes. Recall that a non-trivial FI coupling requires that the D-instantons are absorbed into the D3-branes and we are forced onto the Higgs branch. When the D3-branes are separated, the D-instantons have a choice of which D3-brane to live on. So we expect that a given contribution will correspond to a partition of the k D-instanton over the N separated D3-branes. On a given brane the moduli space will be that of an abelian instanton theory. This is exactly the picture that (4.20) embodies.

We feel optimistic that the localization means that the partition function can be evaluated in terms of a sum over contributions from the individual branches (4.20). In the present paper, rather than present the analysis of the general contribution, we will settle for considering that from the trivial partition $k\to\{k\}$, i.e. the situation when all the D-instantons live on the same D3-brane. There are N such configurations. In order to evaluate them we have to consider the fluctuations around the solution $\widehat{\mathfrak{M}}_{k,1}$. As before, the kinetic terms for the vector multiplet variables χ_a and $\lambda^{\dot{\alpha}}_A$ are higher order in the coupling λ and play no rôle. In addition, as we have previously established, integrating out the fluctuations of the ADHM variables and $\lambda^{\dot\alpha}_A$ produce the explicit δ-functions that impose the ADHM constraints and their Grassmann analogues. Moreover, the χ_a integral produces the factor of $(\det_{k^2}\mathbf{L})^{-3}$. The new ingredient is that the fluctuations q_{iu} and $\tilde q_{ui}$, for $u\neq u_i$, and their fermionic partners receive a mass $|\varphi_{u_i}-\varphi_{u_j}|$. The simplifying feature of this contribution is that these fluctuations decouple from the ADHM constraints at linear order and so the integrals are simply unconstrained gaussians. As expected the determinant factors cancel between the bosonic and Grassmann integrals. We are then left with an integral on the solution space $\widehat{\mathfrak{M}}_{k,1}$, which is simply the original integral with all the variables orthogonal to the solution space set to zero. Therefore, these N contributions are simply abelian instanton partition functions, giving an overall contribution

$$N\widehat{\mathcal{Z}}_{k,1}(\zeta,g_0)=N\widehat{\mathcal{Z}}_{k,1}(\zeta,\infty)\,, \tag{4.21}$$

to $\widehat{\mathcal{Z}}_{k,N}(\zeta,g_0,\varphi)$. This is a remarkable result because we have reduced the problem of calculating part of the multi-instanton partition function in a non-abelian gauge group to one involving an abelian gauge group. This latter quantity (1.12) has been calculated in [24] and so the contribution is

$$N\sum_{d|k}\frac{1}{d}\,. \tag{4.22}$$

We suspect that the other contributions (3.2) for more general partitions can also be calculated, however, the fluctuation analysis is more complicated in these cases because the fluctuations couple the different abelian instanton factors and do not decouple from the ADHM constraints. We shall leave this more general analysis for the future.

If we take $\varphi=0$, then the solution spaces change discontinuously from (4.20) to $\widehat{\mathfrak{M}}^{(\zeta)}_{k,N}$ and we are not surprised to find that the partition function changes also changes discontinuously: compare (2.19) and (4.22) for $k=1$.

In all the cases considered in this section we have had a non-trivial FI coupling which forces us onto the Higgs branch. On the contrary, when $\zeta=0$ the situation is more subtle. The reason is that action is also zero when an instanton shrinks to zero size. In this case the leading order analysis breaks down since at points where $\mathrm{U}(k)$ does not act freely, the operator $\mathbf{L}$ has null eigenvector(s) and the components of χ_a proportional to the null eigenvector(s) are not lifted at gaussian order. The resulting integral is still convergent, however, due to the kinetic term for the vector multiplet variables. In fact, we have seen in the one-instanton sector that keeping g_0 finite has the same effect as the FI coupling of killing the contribution from the small instanton singularity and we would like to argue that this is true for $k>1$. The question is whether there is any discontinuity at $\zeta=0$ with g_0 finite. We strongly believe that there is no such discontinuity and

$$\widehat{\mathcal{Z}}_{k,N}(0,g_0,\varphi)=\widehat{\mathcal{Z}}_{k,N}(\zeta,\infty,\varphi)=\widehat{\mathcal{Z}}_{k,N}(\zeta,g_0,\varphi)\,. \tag{4.23}$$

The reason is that taking $\zeta\to 0$ does not change the long-distance behaviour of the potential (unlike the situation when the VEVs are turned on). One piece of evidence for this is the abelian case $N=1$. In that case, we argued [24]

$$\widehat{\mathcal{Z}}_{k,1}(\zeta,\infty)=\sum_{d|k}\frac{1}{d}\,. \tag{4.24}$$

However, Green and Gutperle [21] have considered certain terms in the D3-brane effective action that are due to D-instanton effects. Based on the S-duality of type-IIB string theory they were led to

$$\widehat{\mathcal{Z}}_{k,1}(0,g_0)=\sum_{d|k}\frac{1}{d}\,. \tag{4.25}$$

Finally, consider the partition function $\widehat{\mathcal{Z}}_{k,N}(0,\infty,\varphi)$. In this case, we have to re-consider the localization on the zeros of the potential (4.13). Since there is no FI coupling, we will also have contributions from the mixed and Coulomb branches. These correspond to points where instantons shrink down to zero size and can move off the D3-branes as D-instantons. The subspace of $\mathfrak{M}_{k,N}$ where n instantons have shrunk to zero size, correspond to the boundaries of the compactification of the instanton moduli space

$$\mathrm{Sym}_n(\mathbb{R}^4)\times\mathfrak{M}_{k-n,N}\,, \tag{4.26}$$

considered in [36]. The calculation of the contributions from these branches to the partition function is complicated; however, just as in the one-instanton sector there is a trick. Since $\widehat{\mathcal{Z}}_{k,N}(0,\infty,0)$ vanishes the contribution from the mixed and Coulomb branches must cancel that from the Higgs branch. Hence, we are led to conjecture that the contribution from the other branches is simply minus that of the Higgs branch $-\widehat{\mathcal{Z}}_{k,N}(\zeta,\infty,0)$. Following on from this is a generalization of (2.50) and (2.51) to $k>1$:

$$\widehat{\mathcal{Z}}_{k,N}(0,\infty,\varphi)=\widehat{\mathcal{Z}}_{k,N}(\zeta,\infty,\varphi)-\widehat{\mathcal{Z}}_{k,N}(\zeta,\infty,0)\,. \tag{4.27}$$

5. The Euler characteristic of the one-instanton moduli space

The result $\widehat{\mathcal{Z}}_{1,N}(\zeta,\infty,\varphi)=N$ has a topological interpretation as the Euler characteristic of the moduli space. This requires some explanation. Firstly, consider the partition function $\widehat{\mathcal{Z}}_{k,N}(\zeta,\infty,0)$. This has the topological interpretation as the

Gauss-Bonnet-Chern (GBC) integral for the manifold $\widehat{\mathfrak{M}}_{k,N}^{(\zeta)}$. For a compact manifold this would give the Euler characteristic. However, the resolved instanton moduli space is non compact because instantons can separate in $\mathbb{R}^4$, as well as grow indefinitely in size. For a non-compact space, one way to define the Euler characteristic is to cut the space off explicitly, giving a compact manifold with boundary on which the Euler characteristic can be defined in the standard way. For the one-instanton moduli space this could be achieved by simply demanding

$$\rho \leq R\,, \tag{5.1}$$

for a large parameter R. Taking $R \to \infty$, the Euler characteristic can then be expressed as the sum of a bulk contribution, given by the GBC integral, and a specific boundary contribution involving the integral of the second fundamental form. One way to calculate the Euler characteristic more directly is to use Morse theory. For the resolved instanton moduli space this has been done by Nakajima [37]. Choosing $\zeta_{\mathbb{C}} = 0$ and $\zeta_{\mathbb{R}} > 0$, the Morse function corresponds to the moment map of a U(1) action on the moduli space given by

$$q \longrightarrow e^{it} q\,, \qquad \tilde{q} \longrightarrow e^{it}\tilde{q}\,, \qquad x \longrightarrow e^{it} x\,, \qquad \tilde{x} \longrightarrow e^{it}\tilde{x}\,. \tag{5.2}$$

It is easy to see (1.11) that this leaves invariant the real ADHM constraint and rotates the complex constraint by a phase. Critical points of the Morse function are then fixed points of the U(1) action. Recalling that the hyper-Kähler quotient construction involves modding out by a U(k) action, we see that critical points are solutions of

$$U_t q = e^{it} q\,, \qquad \tilde{q} U_t^{-1} = e^{it}\tilde{q}\,, \qquad U_t x U_t^{-1} = e^{it} x\,, \qquad U_t \tilde{x} U_t^{-1} = e^{it}\tilde{x}\,, \tag{5.3}$$

where $U_t \in \mathrm{U}(k)$, or infinitesimally

$$\phi q = q\,, \qquad \tilde{q}\phi = -\tilde{q}\,, \qquad [\phi, x] = x\,, \qquad [\phi\, \tilde{x}] = \tilde{x}\,, \tag{5.4}$$

for ϕ in the Lie algebra of U(k).

For $k = 1$, since $\zeta_{\mathbb{R}} > 0$, the critical point set is simply given by $\tilde{q} = 0$ and $qq^\dagger = \zeta_{\mathbb{R}}$, modulo U(1). In other words, the fixed-point set is the $\mathbb{CP}^{N-1}$ that arises from blowing-up the singularity. In particular,

$$\chi\left(\widehat{\mathfrak{M}}_{1,N}^{(\zeta)}\right) = \chi\left(\mathbb{CP}^{N-1}\right) = N\,. \tag{5.5}$$

We now remark that (5.5) is equal $\widehat{\mathfrak{Z}}_{1,N}(\zeta, \infty, \varphi)$. The connection is the following. In our cohomological interpretation of the partition function coupling to the VEVs corresponds to introducing a Morse function on the moduli space corresponding to moment map for the $U(1)^N$ subgroup of the U(N) action on the moduli space.

Upon taking the $\lambda \to 0$ limit, we localize the partition function on the submanifold described by solutions of (4.13) and (1.11) for $k=1$, i.e.

$$qq^\dagger - \tilde{q}^\dagger\tilde{q} = \zeta_{\mathbb{R}}\,, \qquad q\tilde{q} = 0\,, \qquad [\chi_a, \chi_b] = \chi_a q - q\varphi_a = \tilde{q}\chi_a - \varphi_a\tilde{q} = 0\,. \tag{5.6}$$

The conditions (5.6), imply that the solution correspond to fixed points of the $U(1)^N$ subgroup of the flavour symmetry. The fixed-point set consists of the discrete points $\tilde{q} = 0$ and $q_v = \sqrt{\zeta_{\mathbb{R}}}\delta_{uv}$, $u = 1, \ldots, N$, up to U(1) gauge transformations. So the VEVs actually correspond to introducing a more refined Morse function on the moduli space which picks out isolated points on $\mathbb{CP}^{N-1}$.

For $k>1$, as we have seen, the fixed-point set of the $U(1)^N$ action consists of non-compact manifolds involving products of abelian instanton moduli spaces (4.20). Consequently, the value of $\widehat{\mathcal{Z}}_{k,N}(\zeta, \infty, \varphi)$ is not a topological index, as is evident in (4.22).

6. Discussion

In this paper we have considered the D-instanton partition function. In the one instanton sector we have presented explicit calculations which provided a strong hint that some form of localization was at work. This led to us formulate the D-instanton matrix theory in the context of cohomological field theory in $(0+0)$ dimensions. We found that the supersymmetry of the matrix theory naturally implied the existence of a nilpotent fermionic symmetry and, furthermore, the matrix theory action was $\mathcal{Q}$-exact. This analysis enabled us to identify contributions to the partition function in terms of the phase structure of the higher dimensional Dp/D$(p+4)$-brane system. Some of these contributions could be evaluated exactly. It is clear that we have only taken a small step in exploiting the power of this formalism.

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A. Normalization of the instanton measure

The D-instanton integration measure in the $\mathcal{N}=4$ supersymmetric U(N) gauge theory has the following form [4]:

$$\begin{aligned}\mathcal{Z}_{k,N}(\zeta, g_0, \varphi) = {}& \frac{g^4 2^{-k^2/2}\,\pi^{-14k^2}\,(C_1'')^k}{\text{Vol}\,U(k)} \times \\ & \times \int d^{4k^2}a'\, d^{8k^2}\mathcal{M}'\, d^{6k^2}\chi\, d^{8k^2}\lambda\, d^{3k^2}D\, d^{2kN} \times \\ & \times w\, d^{2kN}\bar{w}\, d^{4kN}\mu\, d^{4kN}\bar{\mu}\;\; \exp[-S_{k,N}]\,,\end{aligned} \tag{A.1}$$

where g is the gauge coupling of the four-dimensional gauge theory and $S_{k,N} = g_0^{-2} S_G + S_K + S_D$ with

$$\begin{aligned}
S_G &= \mathrm{tr}_k \Big(-[\chi_a, \chi_b]^2 + \sqrt{2} i\pi \lambda_{\dot\alpha A}[\chi^\dagger_{AB}, \lambda^{\dot\alpha}_B] + 2D^c D^c \Big)\,, \\
S_K &= \mathrm{tr}_k \Big(-[\chi_a, a'_n]^2 + \tilde\chi_a \bar w^{\dot\alpha} w_{\dot\alpha} \tilde\chi_a + \sqrt{2} i\pi \mathcal{M}'^{\alpha A}[\chi_{AB}, \mathcal{M}'^B_\alpha] + 2\sqrt{2} i\pi \bar\mu^A_u \tilde\chi_{AB} \mu^B_u \Big)\,, \\
S_D &= i\pi\, \mathrm{tr}_k \Big([a'_{\alpha\dot\alpha}, \mathcal{M}'^{\alpha A}]\lambda^{\dot\alpha}_A + \bar\mu^A_u w_{u\dot\alpha}\lambda^{\dot\alpha}_A + \bar w_{\dot\alpha}\mu^A \lambda^{\dot\alpha}_A \Big) + \\
&\quad + i\, \mathrm{tr}_k \Big(D^c \Big((\tau^c)^{\dot\beta}{}_{\dot\alpha} (\bar w^{\dot\alpha} w_{\dot\beta} + \bar a'^{\dot\alpha\alpha} a'_{\alpha\dot\beta}) - \zeta^c \Big) \Big)\,. \qquad \text{(A.2)}
\end{aligned}$$

In the above, the 6-vector quantity $\tilde\chi_a$ includes the coupling to the VEVs φ_{au}:

$$(\tilde\chi_a)_{ij,uv} = (\chi_a)_{ij}\delta_{uv} - \delta_{ij}\varphi_{au}\delta_{uv}\,. \qquad \text{(A.3)}$$

We use the same conventions[12] as in [4]. In particular, our normalization convention for integrating Grassmann Weyl spinors is $\int d^2\lambda\, \lambda^2 = 2$, rather than 1. The integrals over the $k \times k$ matrices a'_n, $\mathcal{M}'^A$, χ_a, $\lambda^{\dot\alpha}_A$ and D^c are defined as the integrals over the components with respect to a hermitean basis of $k \times k$ matrices T^r normalized so that $\mathrm{tr}_k\, T^r T^s = \delta^{rs}$. The constant C''_1 was derived in ref. [4, eq. (4.5)] by comparing (A.1) with the one-instanton Bernard measure [38] suitably generalized to an $\mathcal{N} = 4$ theory:

$$C''_1 = 2^{-2N+1/2}\pi^{-6N} g^{4N}\,. \qquad \text{(A.4)}$$

Remaining numerical factors (of 2 and π) on the r.h.s. of (A.1) follow from [4, eqs. (4.1), (4.7) and (4.8)].

The centered partition function $\widehat{\mathcal{Z}}_{k,N}(\zeta, g_0, \varphi)$ is defined by modding out the center of mass bosonic degrees of freedom together with their fermionic superpartners

$$\widehat{\mathcal{Z}}_{k,N}(\zeta, g_0, \varphi) = \frac{\mathcal{Z}_{k,N}(\zeta, g_0, \varphi)}{\int d^4 a'/(2\pi)^2\, [8\pi^2 k/g^2]^2 \int d^8\mathcal{M}'\, [2\pi^2 k/g^2]^{-4}}\,. \qquad \text{(A.5)}$$

The factors of $8\pi^2 k/g^2$ and $2\pi^2 k/g^2$ account for the normalization jacobians of the zero modes (bosonic and fermionic) associated with the overall translations collective coordinates a'_m, and the supersymmetric collective coordinates $\mathcal{M}'^A_\alpha$.

Combining the expressions above we obtain the desired formula for the centered partition function:

$$\begin{aligned}
\widehat{\mathcal{Z}}_{k,N}(\zeta, g_0, \varphi) &= \frac{k^2\, 2^{-k^2/2+k/2-2Nk}\, \pi^{-14k^2-6Nk+6}}{\mathrm{Vol}\,\mathrm{U}(k)} \times \\
&\quad \times \int d^{4(k^2-1)}a'\, d^{8(k^2-1)}\mathcal{M}'\, d^{6k^2}\chi\, d^{8k^2}\lambda\, d^{3k^2}D\, d^{2kN} \times \\
&\quad \times w\, d^{2kN}\bar w\, d^{4kN}\mu\, d^{4kN}\bar\mu\ \exp[-S_{k,N}]\,. \qquad \text{(A.6)}
\end{aligned}$$

[12]The index assignment is $i, j, \ldots = 1, \ldots, k$ and $u, v, \ldots = 1, \ldots, N$, together with $a, b, \ldots = 1, \ldots, 6$ and $c = 1, \ldots, 3$.

References

[1] M.F. Atiyah, N.J. Hitchin, V.G. Drinfeld and Y.I. Manin, *Construction of instantons*, *Phys. Lett.* **A 65** (1978) 185.

[2] E. Corrigan and P. Goddard, *Construction of instanton and monopole solutions and reciprocity*, *Ann. Phys. (NY)* **154** (1984) 253.

[3] V.V. Khoze, M.P. Mattis and M.J. Slater, *The instanton hunter's guide to supersymmetric* SU(N) *gauge theory*, *Nucl. Phys.* **B 536** (1998) 69 [hep-th/9804009].

[4] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instanton calculus and the AdS/CFT correspondence in $N = 4$ superconformal field theory*, *Nucl. Phys.* **B 552** (1999) 88 [hep-th/9901128].

[5] E. Witten, *Small instantons in string theory*, *Nucl. Phys.* **B 460** (1996) 541 [hep-th/9511030].

[6] M.R. Douglas, *Branes within branes*, in *Cargese 1997, Strings, branes and dualities*, hep-th/9512077.

[7] M.R. Douglas, *Gauge fields and D-branes*, *J. Geom. Phys.* **28** (1998) 255 [hep-th/9604198].

[8] N. Dorey, T.J. Hollowood, V.V. Khoze, M.P. Mattis and S. Vandoren, *Multi-instantons and Maldacena's conjecture*, *J. High Energy Phys.* **06** (1999) 023 [hep-th/9810243].

[9] N.J. Hitchin, A. Karlhede, U. Lindstrom and M. Rocek, *Hyperkahler metrics and supersymmetry*, *Comm. Math. Phys.* **108** (1987) 535.

[10] E.F. Corrigan, D.B. Fairlie, S. Templeton and P. Goddard, *A Green's function for the general selfdual gauge field*, *Nucl. Phys.* **B 140** (1978) 31.

[11] N.H. Christ, E.J. Weinberg and N.K. Stanton, *General self-dual Yang-Mills solutions*, *Phys. Rev.* **D 18** (1978) 2013.

[12] E. Witten, *Sigma models and the ADHM construction of instantons*, *J. Geom. Phys.* **15** (1995) 215 [hep-th/9410052].

[13] O. Aharony, M. Berkooz, S. Kachru, N. Seiberg and E. Silverstein, *Matrix description of interacting theories in six dimensions*, *Adv. Theor. Math. Phys.* **1** (1998) 148 [hep-th/9707079].

[14] O. Aharony, M. Berkooz and N. Seiberg, *Light-cone description of* $(2,0)$ *superconformal theories in six dimensions*, *Adv. Theor. Math. Phys.* **2** (1998) 119 [hep-th/9712117].

[15] O. Aharony, *A brief review of 'little string theories'*, *Class. and Quant. Grav.* **17** (2000) 929 [hep-th/9911147].

[16] E. Witten, *On the conformal field theory of the Higgs branch*, *J. High Energy Phys.* **07** (1997) 003 [hep-th/9707093].

[17] M. Berkooz and H. Verlinde, *Matrix theory, AdS/CFT and Higgs-coulomb equivalence*, *J. High Energy Phys.* **11** (1999) 037 [hep-th/9907100].

[18] O. Aharony and M. Berkooz, *IR dynamics of* $D = 2$, $N = (4,4)$ *gauge theories and DLCQ of 'little string theories'*, *J. High Energy Phys.* **10** (1999) 030 [hep-th/9909101].

[19] N. Nekrasov and A. Schwarz, *Instantons on noncommutative* R^4 *and* $(2,0)$ *superconformal six dimensional theory*, *Comm. Math. Phys.* **198** (1998) 689 [hep-th/9802068].

[20] N. Seiberg and E. Witten, *String theory and noncommutative geometry*, *J. High Energy Phys.* **09** (1999) 032 [hep-th/9908142].

[21] M.B. Green and M. Gutperle, *D-instanton induced interactions on a D3-brane*, *J. High Energy Phys.* **02** (2000) 014 [hep-th/0002011].

[22] P. Yi, *Witten index and threshold bound states of D-branes*, *Nucl. Phys.* **B 505** (1997) 307 [hep-th/9704098].

[23] S. Sethi and M. Stern, *D-brane bound states redux*, *Comm. Math. Phys.* **194** (1998) 675 [hep-th/9705046].

[24] N. Dorey, T.J. Hollowood and V.V. Khoze, *Notes on brane bound-state problems, index theory, instantons and monopoles*, to appear.

[25] M.B. Green and M. Gutperle, *D-particle bound states and the D-instanton measure*, *J. High Energy Phys.* **01** (1998) 005 [hep-th/9711107].

[26] T. Banks, *Tasi lectures on matrix theory*, hep-th/9911068.

[27] E. Witten, *Supersymmetry and Morse theory*, *J. Diff. Geom.* **17** (1982) 661.

[28] K. Lee, D. Tong and S. Yi, *The moduli space of two* U(1) *instantons on noncommutative* R^4 *and* $R^3 \times S^1$, *Phys. Rev.* **D 63** (2001) 065017 [hep-th/0008092].

[29] T. Eguchi, P.B. Gilkey and A.J. Hanson, *Gravitation, gauge theories and differential geometry*, *Phys. Rep.* **66** (1980) 213.

[30] K. Lee and P. Yi, *Quantum spectrum of instanton solitons in five dimensional noncommutative* U(N) *theories*, *Phys. Rev.* **D 61** (2000) 125015 [hep-th/9911186].

[31] G. Moore, N. Nekrasov and S. Shatashvili, *D-particle bound states and generalized instantons*, *Comm. Math. Phys.* **209** (2000) 77 [hep-th/9803265].

[32] C. Vafa and E. Witten, *A strong coupling test of s duality*, *Nucl. Phys.* **B 431** (1994) 3 [hep-th/9408074].

[33] D. Bellisai, F. Fucito, A. Tanzini and G. Travaglini, *Instanton calculus, topological field theories and $N = 2$ super Yang-Mills theories*, *J. High Energy Phys.* **07** (2000) 017 [hep-th/0003272].

[34] E. Witten, *Phases of $N = 2$ theories in two dimensions*, *Nucl. Phys.* **B 403** (1993) 159 [hep-th/9301042].

[35] E. Witten, *Topological quantum field theory*, *Comm. Math. Phys.* **117** (1988) 353.

[36] S.K. Donaldson and P.B. Kronheimer, *The geometry of four-manifolds*, Oxford University Press, 1990.

[37] H. Nakajima, *Instantons and affine lie algebras*, *Nucl. Phys.* **46** *(Proc. Suppl.)* (1996) 154 [alg-geom/9510003];
Heisenberg algebra and Hilbert schemes of points on projective surfaces, alg-geom/9507012;
Resolution of moduli spaces of ideal instantons on R^4, in *Sanda 1993, Topology, geometry and field theory* p. 129–136.

[38] C. Bernard, *Gauge zero modes, instanton determinants and quantum-chromodynamic calculations*, *Phys. Rev.* **D 19** (1979) 3013.